Correcting simulation bias in discrete monitoring of Russian options

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ABSTRACT: This contribution analyzes Russian options which are exotic American-style perpetual options for which a valuation formula is available in closed form. It is interesting to compute not only the option price but also the expected exercise time, which is not known in advance. In order to compute this optimal optimal stopping time we apply three different approaches: a crude Monte Carlo method, a simulation algorithm which corrects the simulation for discrete monitoring using a Brownian bridge technique and an extrapolation technique.

Moreover, we tackled a second valuation problem. In the real world options monitoring takes place at discrete times; since the valuation formula for Russian options is obtained assuming a continuous monitoring, by using this formula to value a discretely monitored option we introduce a bias. By applying a crude Monte Carlo simulation method we easily obtain an estimate for the value of such a discrete monitored option. In addition we introduce a correction which allows to use the continuous time valuation formula to approximate the value of a discretely monitored option.

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1. INTRODUCTION

American-style options are among the path-dependent derivatives which are most difficult to value. Let us observe that even for standard American-style options in the classical Black and Scholes environment a closed form valuation formula is not known.

In order to overcome these difficulties, a number of different numerical approaches have been advanced. Such numerical approximations have lead to satisfactory results for standard options but for some exotic options the research on this field is still going on. For example, to value barrier and lookback options we can use lattice methods, but these approaches are not applicable to American-style options written on many underlying assets (where “many” may mean just three or five).

A general approach which can often be used to value exotic options for which analytical formulae are not available is Monte Carlo simulation. However, only recently suitable simulation algorithms have been proposed in the literature for American options (see for example Boyle, Broadie and Glasserman, 1997).

On the other hand, when we apply a simulation approach we implicitly discretize the time horizon since with such a technique we can only monitor the prices at discrete points in time. However, it can be shown that discrete monitoring entails a valuation bias for any non zero simulation interval (see Broadie, Glasserman and Kuo, 1997). Of course, by restricting the monitoring interval between successive observations we may shrink the bias, but which interval is needed to obtain a sufficiently precise valuation? And can we support this additional computational effort?

In this contribution we consider Russian options which are exotic American-style options first studied by Shepp and Shiryaev (1993). These options are perpetual and this feature allows to obtain the valuation formula in closed form. Of course, it is interesting to determine not only the option price but also the expected optimal stopping time: is the expected exercise time about a few weeks or tenths of years? Unfortunately, such information cannot be obtained analytically.

In order to compute this optimal exercise time we have applied three different approaches: a crude Monte Carlo method, a simulation algorithm which corrects simulation for discrete monitoring using a Brownian bridge technique, and an extrapolation technique.

Moreover, we have tackled a second valuation problem: in the real world the options contracts usually state that monitoring is carried out at discrete times (for example each day at ten o’ clock). The problem which arises is therefore how to value such options. In fact, the valuation formula for Russian options is obtained assuming a continuous monitoring. Again, by using this formula to value a discretely monitored option we introduce a bias.

By applying a crude Monte Carlo simulation method we easily obtain an estimate for the value of such a discrete monitored option. In addition, following Broadie, Glasserman and Kuo (1999) we have introduced a correction for the Russian option value which allows to use the continuous time valuation formula to approximate the value of a discretely monitored option with analogous characteristics.

The numerical methods developed have been tested through a wide experimental analysis which shows the applicability of the algorithms proposed.

The paper is organized as follows. Section 2 introduces Russian options and their analytical valuation formula. In Section 3 we analyze the problems encountered by simulating American-style perpetual options. In Section 4 we discuss the bias introduced in the valuation of path-dependent options with discrete monitoring. Section 5 presents three different numerical algorithms advanced for correcting the discrete monitoring bias of Russian options. In Section 6 we report the results of the numerical experiences carried out. Finally, some concluding remarks are presented.

2. RUSSIAN OPTIONS

A Russian option is a perpetual American-style option which, at any time chosen by the holder as stopping time, pays out the maximum realised asset price up to that date.

This option was first studied by Shepp and Shiryaev (1993) and Duffie and Harrison (1993) under the usual assumptions of Brownian motion price dynamics and continuous time sampling. In particular, Shepp and Shiryaev give the analytical formula for the option value by determining the optimal stopping strategy; Duffie and Harrison, on the other hand, using arbitrage arguments, derive the main properties of Russian options.

Let us assume that, in a frictionless market, the dynamics of the underlying asset price $S = \{ S_t, t \geq 0 \}$ is described by the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW, \qquad S_0 > 0 \tag{2.1}$$

where $W = \{ W_t, t \geq 0 \}$ is a standard Brownian motion and the drift μ and the volatility σ are assumed known. Let us further assume that the asset pays dividends continuously at a constant rate $d \geq 0$ proportional to the price S_t.

Let us define the "maximum" stochastic process

$$M = \{ M_t, t \geq 0 \} = \max \left[M_0, \max_{0 \leq u \leq t} S_u \right] \tag{2.2}$$

where $M_0 \geq S_0$ is a starting value for the maximum process. If the holder choose to exercise the option at time $\tau \geq 0$, he receives the payoff M_τ (note that no exercise price is required).

Shepp and Shiryaev (1993) show that if $d = 0$, i.e. if the asset does not pay any dividend, the Russian option has infinite rational value and there is no finite optimal stopping time. On the other hand, Duffie and Harrison (1993) show that the arbitrage Russian option value does not depend on the drift μ; this means that we can use a risk-neutral argument to find out the optimal strategy. In particular, by redefining $\mu = r - d$, where r denotes the instantaneous riskless rate of return, we obtain for the asset dynamics a risk-neutral martingale measure.

Under this risk-neutral measure the price at time $t = 0$ of the Russian option, V_0, is the supremum of the expected discounted payoff over all the stopping times $\tau \geq 0$

$$V_0 = \sup_{\tau \geq 0} E[e^{-r\tau} M_\tau]. \tag{2.3}$$

The fair option price, therefore, can be obtained by solving a problem of optimal stopping time. Shepp and Shiryaev (1993) solve this optimal stopping problem by finding out the optimal stopping rule; at any time $t \geq 0$ the holder can use this rule to test if it is optimal to exercise the option or to continue with it.

Shepp and Shiryaev's option valuation formula is

$$V_0 = \begin{cases} M_0 & S_0 \leq M_0/\alpha \\ \dfrac{M_0}{\gamma_2 - \gamma_1} \left[\gamma_2 \left(\dfrac{\alpha S_0}{M_0} \right)^{\gamma_1} - \gamma_1 \left(\dfrac{\alpha S_0}{M_0} \right)^{\gamma_2} \right] & M_0/\alpha < S_0 \end{cases} \tag{2.4}$$

where:

$$\alpha = \left(\frac{1 - \gamma_1^{-1}}{1 - \gamma_2^{-1}} \right)^{(\gamma_2 - \gamma_1)^{-1}}, \tag{2.5}$$

$$\gamma_{1,2} = \frac{1}{2} - \frac{r-d}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} - \frac{r-d}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}}, \tag{2.6}$$

with $\gamma_1 < 0 < 1 < \gamma_2$, and the optimal stopping time τ^* is the first t for which $S_t \leq M_t/\alpha$

$$\tau^* = \inf \{ t \geq 0 : S_t \leq M_t/\alpha \}. \tag{2.7}$$

Note that the stopping criterion (2.7) gives a practical rule to decide whether it is optimal to immediately exercise the option, but it does not allow to compute in advance the path-dependent optimal stopping time τ^*: the value of τ^* will depend on the ratio between the future values of S_t and M_t.

It is known that Russian options are not dealt with in the official financial markets; however, the stopping time problems studied for these options may have interesting financial and actuarial applications. For instance, an actuarial application of the Russian option valuation is proposed by Gerber and Pafumi (1998). These authors analyze the guaranteed value of an investment fund with a particular embedded exotic protection; this protection guarantees that the fund value at any time t will not be lower than a fixed proportion of the maximum value of this fund, observed up to time t. They show that the price of the guarantee can be written as the product of two terms: the value of an associated Russian option and a term which depends only on the guaranteed proportion.

3. SIMULATING AMERICAN-STYLE PERPETUAL OPTIONS

We have seen that formulae (2.4)–(2.6) allow to value a Russian option in closed form. However, the associated formula (2.7) gives an optimal stopping rule but does not allow to compute the expected optimal exercise time of the option. In particular, this formula gives the holder no idea about the expected optimal duration of his investment: the average duration could be one month as well as ten years!

To study this optimal stopping time we can use Monte Carlo techniques which permit to analyze other interesting features, besides the derivative value.

As it is known, Monte Carlo techniques works building simulated paths for the underlying security price which are computed recursively. Under the geometric Brownian motion assumption, the price paths are generated as follows

$$S_{i\Delta t} = S_{(i-1)\Delta t}\, e^{(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} z_i} \qquad i = 1, 2, \ldots \tag{3.1}$$

where Δt is the length of the time interval between two successive "observed" prices and z_i are random numbers drawn from a standard gaussian distribution.

Monte Carlo simulation procedures generate the random numbers z_i as independent pseudo-random deviates. In practice, these variates are calculated with a two step procedure: first a uniform pseudo random number u_i is drawn in the interval $(0, 1)$ and then u_i is transformed (with a convenient algorithm) into a corresponding gaussian deviate z_i.

Monte Carlo procedures require to generate M price paths and take the average value of the discounted option payoffs thus obtained as the estimation of the option price. Let us denote by τ_m the optimal stopping time along path m and by V_m the option payoff obtained exercising the option at time $t = \tau_m$; then the Monte Carlo estimate of the option value V is

$$\hat{V} = \frac{1}{M} \sum_{m=1}^{M} e^{-r\tau_m} V_m. \tag{3.2}$$

Analogously, we can estimate the optimal exercise time by computing the average exercise time for all the generated paths

$$\hat{\tau} = \frac{1}{M} \sum_{m=1}^{M} \tau_m. \tag{3.3}$$

Quasi Monte Carlo (or low discrepancy sequences) techniques generate random vectors $(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ which have the property of being "well" distributed in the hypercube $(0,1)^n$; then the vector elements u_i have to be transformed into the corresponding gaussian deviates z_i by a suitable algorithm.

Recently, many experimental results (see e.g. Joy, Boyle and Tan, 1996, and Galanti and Jung, 1997) have shown that Quasi Monte Carlo procedures perform better than simulation techniques for many exotic options. However, low discrepancy sequences present two features which make them not suitable, at least directly, for American-style perpetual options.

A first drawback is due to the fact that the exercise time is not known in advance and therefore the dimension n of the vectors that have to be generated differs from a path to another, and can even be not finite. Of course, this inconvenient could be often overcome by choosing a very large value for n, but the values required for n can be very large and may cause serious multicollinearity problems.

A second serious drawback is caused by the monotonicity of many sequences generated by the low discrepancy methods. Think, for example, to the constant vector $(u, u, \ldots, u)$ (with $u \in (0,1)$) which often appears in the low discrepancy sequences; when it is transformed into a gaussian vector, we get a strictly monotone path. If

$$u > u_0 = N(z_0) \qquad \text{with } z_0 = -\frac{\left(r - \sigma^2/2\right)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{3.4}$$

where $N(\cdot)$ is the cumulative distribution function of the standard gaussian random variable, a strictly increasing price path is obtained. If we consider a Russian option (but the assertion holds for other American-style perpetual options, too), we may note that the stopping criterion is never satisfied in such an increasing path; hence, the optimal exercise time is not finite and an infinite value is obtained for the option price.

Obviously, in order to have finite values for the estimates $\hat{V}$ and $\hat{\tau}$, the stopping time $\hat{\tau}_m$ has to be finite for all m. Unfortunately, vectors with constant elements are frequent in the most used Quasi Monte Carlo sequences.

On the other hand, from a theoretical point of view, increasing paths could be generated also by standard simulation techniques. However, nowadays "good" random number generators are available that have overcome all the usual statistical tests of multidimensional independent uniform distribution. With such "good" generators the probability of drawing an increasing path is very low when a sufficiently large number of periods is considered, and tends to zero as this number tends to infinity.

Furthermore, Monte Carlo simulation procedures build a price path by generating the price changes step by step; this feature allows to exploit the knowledge of an optimal stopping rule in order to cease the path simulation as soon as this optimal criterion is met. Indeed, the very absence of an optimal stopping rule for American-style options with a finite expiry date makes simulation much harder for pricing these options; for a review on this subject see for example Boyle, Broadie and Glasserman (1997).

For these reasons we prefer to carry out the simulation of Russian options with standard Monte Carlo techniques.

4. DISCRETE MONITORING OF PATH-DEPENDENT OPTIONS

Let us observe how the path of the underlying asset price is generated in simulation approaches: in formula (3.1) the asset prices are simulated at discretely sampled points, usually equally spaced. However, by proceeding in discrete time we neglect what happens between two adjacent points; in particular, we do not test continuously the optimal stopping rule (2.7).

In fact, by discretely monitoring the underlying asset price we could ignore either the appearance of a new maximum or the fulfillment of the optimal stopping rule, if they occur at any time between two sampled points. It can be seen that discrete monitoring introduces in the option valuation, for any non zero time interval Δt, a bias. Of course, this bias vanishes when the time interval Δt tends to zero but it can cause substantial errors even for relatively close sampled points.

Table 1 shows how this discrete monitoring bias can affect the estimation of both the option value and the average exercise time of a Russian option. The table considers discrete monitoring with monthly, weekly, daily and infra-daily (5

Table 1. Monte Carlo simulation results for a Russian option; the price of the underlying asset is (risk neutrally) lognormally distributed with parameters $S_0 = M_0 = 100$, $\sigma = 0.3$, $r = 0.1$, $d = 0.05$. The simulation results concern 100 000 paths; the standard errors are reported in parentheses; the stopping time is measured in years.

Frequency	Analytic value	Monte Carlo values	Relative errors	Stopping times	Number of periods
12	137.040	128.647 (0.359)	-0.06124	5.076 (0.013)	61
52	137.040	133.379 (0.318)	-0.02671	4.292 (0.011)	223
250	137.040	135.332 (0.295)	-0.01246	3.926 (0.010)	982
1250	137.040	136.085 (0.294)	-0.00697	3.772 (0.010)	4715

points a day) frequencies. The results reported concern 100 000 simulated paths; the Monte Carlo estimates are compared to the analytic option value computed by formula (2.4).

As was expected, the relative error decreases with the sample frequency but even with a daily simulation interval the relative error is over 1 %. By using an infra-day frequency the relative error goes down, but at the cost of computing a higher number of sampled points: note that if we turn from a daily frequency to a 5-times-a-day one, the average number of periods for each simulated path (column 6 of Table 1) is multiplied by almost five. Moreover, the estimation of the average optimal exercise time diminishes heavily with the sample frequency; hence, when a low frequency is used the average optimal exercise time is remarkably overestimated.

Actually, discrete monitoring biases have been observed for other exotic options, too. In particular, this problem is known to occur for two classes of options and precisely for lookback options, which depend on the extreme values of the underlying asset price, and for barrier options, which depend on whether the underlying asset price touches a predetermined level or not.

When the price of the underlying security is observed continuously, closed form valuation formulae are known for these options (see Goldman, Sosin and Gatto, 1979, and Rubinstein and Reiner, 1991). However, the derivative contracts settled in real markets refer to discrete sampling: the dynamics of the underlying security price is observed only at fixed times (for example, once a day at a fixed hour). In these circumstances, the continuous time formulae do not allow to price discretely monitored options accurately; a bias is thus introduced in the valuation.

If we use a simulation technique with a sampling time interval Δt in order to price one of these path-dependent options, we obtain an unbiased estimate for the option whose monitoring interval is equal to Δt. This is the reason why crude simulation cannot give an unbiased estimate of a continuously monitored option.

Some methods have been suggested in the literature either to obtain a more accurate estimation of the value of a discretely monitored option when a continuous monitoring formula is known or to give a more precise estimation of a continuously monitored option using a discrete sampling technique such as simulation.

The first approach is followed by Broadie, Glasserman and Kuo (1997, 1999) who devise some corrections to be applied to the continuous time formulae in order to better estimate the value of discretely sampled lookback and barrier options. These correction terms can be interpreted as shifting the barrier, the strike price or an extremal price. Another numerical technique related to this first approach is proposed by Levy and Mantion (1997) who use the knowledge of both the continuous time formula and the discrete time formulae for one and two monitoring dates; these three valuation formulae are interpolated, using Richardson's formula, to find an approximation for the value of a lookback or barrier option with a higher number of monitoring dates.

The second approach is proposed by Beaglehole, Dybvig and Zhou (1997) and El Babsiri and Noel (1998) to price continuous time lookback, barrier and some other exotic options. Both contributions use a special Monte Carlo simulation technique which, at each sampled time interval, first generates the "observed" final price of the underlying security and then generates the extremal value for this price which has occurred within the interval. In such a way, the appearance of new extremal values between two successive sample dates is explicitly taken into consideration. Another numerical method which falls in this second approach is proposed by Baldi, Caramellino and Iovino (1999) to evaluate single and double barrier options with general features; this method uses Sharp Large Deviations estimates to improve the usual Monte Carlo procedure.

5. CORRECTING THE DISCRETE MONITORING BIAS OF RUSSIAN OPTIONS

Let us come back to the Russian options and the problems connected to their evaluation. We have seen that an analytical formula for the price of Russian options and an optimal stopping rule are known for continuous time monitoring; nevertheless, even in this case a closed form formula is not known for the average optimal stopping time. Moreover, if the continuous time formula is used to value options which refer to discrete sampling, a bias is introduced.

Simulation techniques can be used to overcome these difficulties. However, it is known that simulation is time consuming and therefore it could be worth devising a numerical approximation, such as those proposed in the literature for lookback and barrier options, to value a discretely monitored Russian option. In addition, if we want to use simulation to estimate the average optimal exercise time of a continuously monitored option we have to "correct" in some way its discrete nature.

On the other hand, the methods proposed in the literature for connecting discrete and continuous path-dependent options are not immediately applicable to Russian options but require some modifications. In fact, these methods assume to operate with finite-maturity options while Russian options are perpetual. In particular, some methods consider a fixed number n of subintervals in which the time-to-maturity is subdivided, and it is not clear the meaning of this number in the case of perpetual options.

5.1. A CORRECTION FORMULA

In order to find a numerical approximation which allows to value a discretely monitored Russian option by making some corrections to the continuous time formula, let us remind from the previous section that by discretely monitoring the underlying asset price we could lack observing two facts, if they occur at any time between two sampled points: the appearance of a new maximum and the fulfillment of the optimal stopping rule.

A correction which accounts for the occurrence of a new maximum is proposed by Broadie, Glasserman and Kuo (1999) with reference to the pricing of hindsight options. The correction proposed by these authors is

$$h_n(M, X) = h_0\left(M_0 e^{\beta\sigma\sqrt{\Delta t}}, X e^{\beta\sigma\sqrt{\Delta t}}\right) e^{-\beta\sigma\sqrt{\Delta t}} + o\left(\frac{1}{\sqrt{n}}\right) \tag{5.1}$$

where $h_0(M, X)$ is the continuous time value of a hindsight option with exercise price X when the current value of the maximum is M, $h_n(M, X)$ is the value of the corresponding option with n discretely sampled monitoring dates and

$$\beta = -\frac{\zeta(1/2)}{\sqrt{2\pi}} \simeq 0.5826 \tag{5.2}$$

with ζ the Riemann zeta function. On a probabilistic base, corrections like the one given by equation (5.1) rely on Brownian motion approximations to first passage probabilities (see Siegmund and Yuh, 1982).

Correction (5.1) acts on the continuous time formula at two levels: first the current maximum value is "corrected" by multiplying it by the term $e^{\beta\sigma\sqrt{\Delta t}}$, then the value thus obtained is multiplied by the inverse correction term $e^{-\beta\sigma\sqrt{\Delta t}}$.

With reference to Russian options, if we adopt a similar correction to take into account the discrete monitoring of the maximum, we obtain an approximation for the value of the option with monitoring interval Δt. In the case $M_0/\alpha < S_0$ such approximation is

$$\begin{aligned} V_{\Delta t} &= e^{-\beta\sigma\sqrt{\Delta t}} \left\{ \frac{M_0 e^{\beta\sigma\sqrt{\Delta t}}}{\gamma_2 - \gamma_1} \left[\gamma_2 \left(\frac{\alpha S_0}{M_0 e^{\beta\sigma\sqrt{\Delta t}}} \right)^{\gamma_1} - \gamma_1 \left(\frac{\alpha S_0}{M_0 e^{\beta\sigma\sqrt{\Delta t}}} \right)^{\gamma_2} \right] \right\} \\ &= \frac{M_0}{\gamma_2 - \gamma_1} \left[\gamma_2 \left(\frac{\alpha S_0}{M_0 e^{\beta\sigma\sqrt{\Delta t}}} \right)^{\gamma_1} - \gamma_1 \left(\frac{\alpha S_0}{M_0 e^{\beta\sigma\sqrt{\Delta t}}} \right)^{\gamma_2} \right]; \end{aligned} \tag{5.3}$$

of course, if $M_0/\alpha \geq S_0$ the option is exercised immediately and no correction is needed.

Using differential calculus techniques it can be shown that

$$V_{\Delta t} \leq V_0 \tag{5.4}$$

i.e. the correction value (5.3) is lower that the continuous time value.

As regards the discrete monitoring of the optimal stopping rule, we observe that the stopping criterion in a sense could be interpreted as a "stochastic moving barrier" which separates the continuation region from the exercise region. The value of this stochastic barrier is

$$H_t = \frac{M_t}{\alpha}. \tag{5.5}$$

Broadie, Glasserman and Kuo (1997) suggest the following continuity correction for a barrier option with discrete monitoring

$$B_n(H) = B_0\left(He^{\pm\beta\sigma\sqrt{\Delta t}}\right) + o\left(\frac{1}{\sqrt{n}}\right) \tag{5.6}$$

where $B_0(H)$ is the continuous time value of a barrier option when the barrier is equal to H, $B_n(H)$ is the value of the corresponding option with n discretely sampled monitoring dates and $+$ applies if $H > S_0$ and $-$ applies if $H < S_0$.

Let us assume that the stochastic barrier value for a discretely monitored Russian option can be approximated by a formula which is analogous to the continuous time formula M_t/α. The discrete monitoring causes two different effects on a barrier of this type: on the one hand, the barrier can be hit by the asset price

S_t between two monitoring dates, on the other hand the stochastic barrier value M_t/α is computed using the discretely observed maximum. The two effects would require opposite corrections: to take into account the first effect, we should lower the barrier value (it should be multiplied by the correction term $e^{-\beta\sigma\sqrt{\Delta t}}$), while to correct for the second effect the barrier should be risen (the maximum value in (5.5) should be corrected through the multiplicative term $e^{\beta\sigma\sqrt{\Delta t}}$). Because of the compensation of the two effects, the discrete time barrier should approximately remain the same.

5.2. AN EXTRAPOLATION TECHNIQUE

A different numerical approach which can be used to connect continuous and discrete time phenomena is Richardson extrapolation technique. This technique has been first used in finance in a well-known paper by Geske and Johnson (1984) to evaluate an American put option. The value of the put option is determined by fist calculating the value of analogous options which can be exercised only at maturity T, at $T/2$ and T, at $T/3$, $2T/3$ and T, and then extrapolating the value of the continuous exercise option through the Richardson technique. Along this line, Levy and Mantion (1997) use the continuous exercise time formula for lookback and barrier option, together with the value of the corresponding discrete time options with only one or two monitoring dates, to give an approximation for the value of the analogous option with any number n of monitoring dates.

As proposed by Geske and Johnson, let us denote by $F(\Delta t)$ a real function of the time step Δt and let us assume that F can be written as follows

$$F(\Delta t) = F(0) + a_1(\Delta t)^p + a_2(\Delta t)^r + o((\Delta t)^s), \tag{5.7}$$

where $p < r < s$ and $a_1, a_2 \in \mathbb{R}$. By neglecting the term $o((\Delta t)^s)$, formula (5.7) can be used either to approximate the continuous time formula $F(0)$, as in Geske and Johnson's approach, or to evaluate the discrete time value $F(\Delta t)$ if the continuous time value is known, as in the approach of Levy and Mantion. Note that the function $F(\Delta t)$ can represent an option value but also other interesting quantities related to discrete monitoring option pricing, such as the optimal stopping time.

In order to determine the value of the coefficients a_1, a_2, we can follow Geske and Johnson's approach which suggest to compute formula (5.7) in correspondence with two different time steps $k\Delta t$ and $q\Delta t$ with $q > k > 1$. By omitting the term $o((\Delta t)^s)$, substituting for a_1, a_2 and solving for $F(0)$ we obtain

$$F(0) \simeq F(\Delta t) + \frac{A}{C}[F(\Delta t) - F(k\Delta t)] - \frac{B}{C}[F(k\Delta t) - F(q\Delta t)] \tag{5.8}$$

where:

$$\begin{aligned} A &= q^r - q^p + k^p - k^r \\ B &= k^r - k^p \\ C &= q^r(k^p - 1) - q^p(k^r - 1) + k^r - k^p. \end{aligned} \tag{5.9}$$

As concerns the values of the parameters p and r, Geske and Johnson choose $p = 1$ and $r = 2$; this choice gives the expansion in a Taylor series around $F(0)$ where the terms above the second order are omitted. Levy and Mantion (1997), instead, choose $p = 1/2$ and $r = 1$ on the basis of graphical inspections and the results by Broadie, Glasserman and Kuo (1997, 1999). Hence, the choice of the values of the parameters p and r is not obvious: the values $p = 1$ and $r = 2$ are appealing for the correspondence with the Taylor series expansion, but on the other hand the values $p = 1/2$ and $r = 1$ seem to be suggested by probabilistic considerations (Siegmund and Yuh, 1982).

In order to use this approach for the evaluation of the optimal stopping time of a continuously monitored Russian option, we need a formula for the discrete stopping time for three different time steps. As such discrete time formula is not known, we suggest to approximate the value of $F(\Delta t), F(k\Delta t), F(q\Delta t)$ using Monte Carlo simulation. Therefore the continuous optimal exercise time τ^* can be approximated by

$$\tau^* \simeq \tau_{\Delta t} + \frac{A}{C}[\tau_{\Delta t} - \tau_{k\Delta t}] - \frac{B}{C}[\tau_{k\Delta t} - \tau_{q\Delta t}] \tag{5.10}$$

where $\tau_{\Delta t}$ represent the optimal stopping time for the option monitored at times $\Delta t, 2\Delta t, 3\Delta t, \ldots$.

Of course, by using simulation to evaluate $\tau_{\Delta t}, \tau_{k\Delta t}, \tau_{q\Delta t}$ we obtain an extrapolated value τ^* which is affected by another source of errors, besides the exclusion of the higher order terms of the expansion.

5.3. A CORRECTION OF THE SIMULATION BIAS

Another method to evaluate the continuously monitored optimal stopping time modifies the simulation technique in order to take into account the fact that a new maximum and/or a new minimum may occur between two monitoring dates.

In fact, some recent contributions first generate the price path of the underlying security at the sampling dates $S_{t+\Delta t}, S_{t+2\Delta t}, \ldots$ and then simulate the extreme values for each sub-interval connecting two adjacent dates, conditional on the simulated path. This procedure has been proposed in the literature to evaluate through simulation some exotic options (see Beaglehole, Dybvig and Zhou, 1997, for lookback, barrier and swing options and El Babsiri and Noel, 1998, for complex lookback and barrier options) and regards the brownian bridge simulation.

Briefly, this procedure is made up of two fundamental steps:

- Step 1: the path of the underlying security at the sampling dates $\Delta t, 2\Delta t, \ldots$ is simulated as in (3.1) obtaining the simulated path $\tilde{S}_{\Delta t}, \tilde{S}_{2\Delta t}, \ldots$.
- Step 2: for each time sub-interval connecting two adjacent dates $[(i-1)\Delta t, i\Delta t]$ the extreme values conditional on the initial and terminal prices are simulated.

The extreme value simulation of step 2 is straightforward since the conditional distribution of both the maximum and the minimum is known (see Karatzas and Shreve, 1988):

$$Prob\left\{ \max_{(i-1)\Delta t \le t \le i\Delta t} \log \frac{S_t}{S_{(i-1)\Delta t}} \le \tilde{S}_{max} \middle| \log \frac{S_{i\Delta t}}{\tilde{S}_{(i-1)\Delta t}} = \log \frac{\tilde{S}_{i\Delta t}}{\tilde{S}_{(i-1)\Delta t}} \right\} =$$
$$1 - \exp\left\{ - \frac{2S_{max}\left(\log \frac{\tilde{S}_{i\Delta t}}{\tilde{S}_{(i-1)\Delta t}} - S_{max}\right)}{\sigma^2 \Delta t} \right\}, \tag{5.11}$$

$$Prob\left\{ \min_{(i-1)\Delta t \le t \le i\Delta t} \log \frac{S_t}{S_{(i-1)\Delta t}} \le \tilde{S}_{min} \middle| \log \frac{S_{i\Delta t}}{\tilde{S}_{(i-1)\Delta t}} = \log \frac{\tilde{S}_{i\Delta t}}{\tilde{S}_{(i-1)\Delta t}} \right\} =$$
$$\exp\left\{ - \frac{2S_{min}\left(\log \frac{\tilde{S}_{i\Delta t}}{\tilde{S}_{(i-1)\Delta t}} - S_{min}\right)}{\sigma^2 \Delta t} \right\}. \tag{5.12}$$

When applied to the simulation of a Russian option, the Brownian bridge technique requires to draw for each sampling interval three independent uniform numbers. The first number is used to generate the price S_t at the end of the interval (by means of a numerical algorithm which gives a normal variate to be used in equation (3.1)). The other two uniform numbers are used to generate the maximum and minimum values in the interval, conditional to the terminal values, by simply inverting the probability distribution functions (5.11) and (5.12).

On the one hand, this procedure allows to update the continuous time maximum at the end of each sampling interval. On the other hand, it permits to check at the end of each interval if the optimal stopping rule (2.7) has been satisfied at some time within the interval.

6. APPLICATIONS AND NUMERICAL COMPARISONS

In order to test the goodness of the correction techniques proposed, we have carried out a number of simulation experiments. In all the simulations performed, the continuously compound riskless rate of return is equal to $r = 0.1$ and the current price of the underlying asset is equal to the initial maximum value $S_0 = M_0 = 100$. The trials concern different values of volatility σ and dividend yield d; more precisely, the values considered are $\sigma = 0.1, 0.2, 0.3, 0.4$ and $d = 0.01, 0.05, 0.09$ (remind that we must have $d > 0$ in order to have a finite value for the option). For each combination of the parameters, 100 000 paths of the underlying asset price have been generated; for each path, the stopping rule (2.7) has been tested at each monitoring time. The experiments have been performed with all the correction techniques described in the previous section.

Tables 2 and 3 present the results for the option values and the stopping times obtained with $\sigma = 0.2$ while table 4 reports the average results obtained for all volatility values.

In particular, Table 2 reports the option values given by the various approximation procedures. The first part of this table compares the continuously monitored value of a Russian option given by equation (2.4)–(2.6) with the values obtained using the Brownian bridge correction technique, crude Monte Carlo simulation and the discrete correction formula (5.3). As it can be seen, the Brownian bridge correction tends to overestimate the continuous option value while the crude Monte Carlo method underestimates it. In effect, the Brownian bridge correction is able to make the estimated value closer to the continuous one, but introduces a bias of opposite sign; this bias tends to vanish when the frequency is augmented. On the other hand, the standard deviation of the Brownian bridge estimate is lower than that of the crude Monte Carlo estimate, but the former method requires a number of random number generations which is threefold.

As have been pointed out in Section 4, Monte Carlo simulation method gives an estimate of the value of a discretely monitored option. If we compare the Monte Carlo estimates with the values obtained with the discrete correction (5.3), we can see that these two values are very close and the latter value is always well within the confidence interval of the simulation estimates for the usual confidence levels. Therefore, the correction value (5.3) seems fairly accurate. As regards the bias introduced by discrete monitoring, we can see that for Russian options it has a negative sign and it is not negligible neither for infra daily observations.

The second part of Table 2 analyzes the results obtained with the Richardson extrapolation techniques proposed in Section 5 and shows the option values given

Table 2. Continuous and discrete monitoring values for Russian options with $S_0 = M_0 = 100$, $\sigma = 0.2$, $r = 0.1$. The Richardson extrapolation values concerns the extrapolation technique applied to the Monte Carlo values; the low frequency results are obtained from the monthly, weekly and daily monitoring outcomes; the high frequency results regard the weekly, daily and infra-daily outcomes.

	Freq=12	Freq=52	Freq=250	Freq=1250
d=0.01				
Continuous value	137.746	137.746	137.746	137.746
Brownian bridge value	140.408	140.798	139.327	137.801
(std. dev.)	(0.309)	(0.464)	(0.451)	(0.416)
Monte Carlo value	130.369	135.962	136.587	137.052
(std. dev.)	(0.568)	(0.608)	(0.494)	(0.529)
Corrected discrete value	133.230	135.548	136.737	137.294
d=0.05				
Continuous value	115.913	115.913	115.913	115.913
Brownian bridge value	118.241	117.569	116.987	116.246
(std. dev.)	(0.075)	(0.079)	(0.088)	(0.090)
Monte Carlo value	110.780	113.726	114.957	115.277
(std. dev.)	(0.109)	(0.102)	(0.097)	(0.094)
Corrected discrete value	112.244	114.093	115.070	115.533
d=0.09				
Continuous value	110.914	110.914	110.914	110.914
Brownian bridge value	112.720	112.414	111.917	111.373
(std. dev.)	(0.052)	(0.049)	(0.043)	(0.054)
Monte Carlo value	106.031	108.923	110.075	110.491
(std. dev.)	(0.063)	(0.061)	(0.058)	(0.057)
Corrected discrete value	107.523	109.201	110.113	110.552

	Closed form value	Extrapolation value p=1, r=2 Low freq.	Extrapolation value p=1, r=2 High freq.	Extrapolation value p=1/2, r=1 Low freq.	Extrapolation value p=1/2, r=1 High freq.
			Richardson extrapolation values		
d=0.01	137.746	136.706	137.186	135.983	137.509
d=0.05	115.913	115.314	115.360	115.860	115.420
d=0.09	110.914	110.407	110.604	110.885	110.773

by the two parameters combinations suggested; furthermore, the results concern two different data frequencies. On the whole, the extrapolation values are fairly close to the continuously monitored values. As concerns the parameter choice, the results are similar, even if the couple $p = 1, r = 2$ seems to give less variable

Table 3. Optimal stopping times of Russian options with $S_0 = M_0 = 100$, $\sigma = 0.2$, $r = 0.1$, computed using the Brownian bridge simulation technique, crude Monte Carlo simulation and the Richardson extrapolation approaches.

	Freq=12	Freq=52	Freq=250	Freq=1250
d=0.01				
Brownian bridge time	5.844	8.088	9.38	10.096
(std. dev.)	(0.016)	(0.023)	(0.026)	(0.029)
Monte Carlo time	15.339	12.755	11.591	11.085
(std. dev.)	(0.044)	(0.037)	(0.033)	(0.032)
Extrapolation time for p=1,r=2			11.250	10.944
Extrapolation time for p=1/2,r=1			10.685	10.691
d=0.05				
Brownian bridge time	1.076	1.383	1.698	1.855
(std. dev.)	(0.003)	(0.004)	(0.004)	(0.005)
Monte Carlo time	3.286	2.585	2.263	2.114
(std. dev.)	(0.009)	(0.007)	(0.006)	(0.006)
Extrapolation time for p=1,r=2			2.169	2.072
Extrapolation time for p=1/2,r=1			2.009	1.994
d=0.09				
Brownian bridge time	0.605	0.654	0.815	0.917
(std. dev.)	(0.001)	(0.002)	(0.002)	(0.002)
Monte Carlo time	1.761	1.342	1.147	1.074
(std. dev.)	(0.005)	(0.003)	(0.003)	(0.003)
Extrapolation time for p=1,r=2			1.090	1.054
Extrapolation time for p=1/2,r=1			0.992	1.023

outcomes.

Table 3 exhibits the estimated stopping times corresponding to the Brownian bridge method, crude Monte Carlo simulation and the extrapolation approaches. These stopping times turn out to depend heavily on the dividend yield value. More precisely, they increase with $\mu = r - d$.

From this table we can note that discrete monitoring, as given by the Crude Monte Carlo method, entails an increase in the expected stopping times. Of course, the discrete monitoring stopping times decrease as the monitoring frequency augments and tend to the unknown continuous monitoring value.

On the other hand, The Brownian bridge approach seems to underestimate the continuous monitoring stopping times, and this bias is more remarkable for the low frequency simulations. In fact, the Brownian bridge stopping times heavily

Table 4. Mean relative and absolute errors for the different approximation techniques for Russian options with $S_0 = M_0 = 100$, $r = 0.1$. The mean values are computed using the parameter values $d = 0.01, 0.05, 0.09$ and $\sigma = 0.1, 0.2, 0.3, 0.4$ (the case $d = 0.01, \sigma = 0.4$ is excluded from the mean values since it exhibits very high optimal stopping times and highly variable option values).

	Freq=12	Freq=52	Freq=250	Freq=1250
Mean relative errors				
Brownian bridge vs. continuous	0.0292	0.0209	0.0065	0.0050
Monte Carlo vs. continuous	-0.0541	-0.0216	-0.0143	-0.0044
Monte Carlo vs. discrete correction	-0.0175	-0.0032	-0.0058	-0.0006
Extrapolation vs. continuous for p=1,r=2			-0.0124	-0.0015
Extrapolation vs. continuous for p=1/2,r=1			-0.0128	0.0065
Mean absolute errors				
Brownian bridge vs. continuous	0.0292	0.0209	0.0117	0.0050
Monte Carlo vs. continuous	0.0541	0.0216	0.0143	0.0044
Monte Carlo vs. discrete correction	0.0175	0.0045	0.0058	0.0008
Extrapolation vs. continuous for p=1,r=2			0.0124	0.0037
Extrapolation vs. continuous for p=1/2,r=1			0.0130	0.0085

increase with the data frequency and are always greater than the discrete Monte Carlo values; this gap tends to vanish for the highest frequencies.

The stopping times obtained with the extrapolation techniques are very close each other and seem to give more reliable estimates of the continuous monitoring times. In fact, the stopping times obtained with these extrapolation methods seem give a good approximation of the limit value of both discrete Monte Carlo and Brownian bridge sequences.

Table 4 presents a synthesis of the results obtained with the different approaches for the different parameter values, given in terms of mean relative and absolute errors of the option values. Of course, the Brownian bridge and extrapolation approximations are compared to the continuous monitoring value while the Monte Carlo values are confronted with both the continuous value and the discrete correction approximation.

From Table 4 we can observe that:

i) the discrete correction (5.3) seems fairly good since it is coherent with the discrete monitoring Monte Carlo results;

ii) the Monte Carlo results are always lower than the continuous monitoring values and the bias tends to vanish when the monitoring frequency rises;

iii) the Brownian bridge simulation approach overestimates the true (continuous monitoring) option values;

iv) the Richardson extrapolation technique which gives the more stable results is the one with $p = 1, r = 2$; these results are more accurate than the Brownian bridge outcomes for the highest frequencies.

7. CONCLUDING REMARKS

In this contribution we have focused on two different aspects of Russian options.

A first analysis regards the connections between discrete and continuous monitoring of path-dependent options. To this regard, we have advanced different numerical approaches which allow to approximately value a discretely monitored Russian option.

A second important aspect concerns the expected optimal exercise time of an option which is perpetual. Since this expected optimal stopping time is not known a priori, we have estimated it by crude Monte Carlo simulation, by a Brownian bridge simulation approach and by an extrapolation technique.

The numerical experiments carried out indicate that the Monte Carlo results are always lower than the continuous monitoring values while the Brownian bridge simulation approach overestimates the continuous monitoring option values. Moreover, the results obtained with the Richardson extrapolation technique are more accurate than the Brownian bridge outcomes for the highest frequencies. In addition, the discrete correction method (5.3) seems to give a sufficiently accurate approximation for the value of a discretely monitored option.

As concerns the expected exercise time, the Brownian bridge method seems to underestimate this time while a more accurate approximation is given by the extrapolation technique. Furthermore, discrete monitoring seems to entail an increase in the stopping time.

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