On modal μ -calculus in S5 and applications

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Introduction

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Model theoretic and algorithmic properties of μ -calculus are interesting, both on arbitrary graphs and on subclasses of graphs.

In this talk we will consider two important subclasses of graphs, S5 and K4.

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Preliminaries

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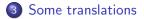


2 Model checking and satisfiability

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2 Model checking and satisfiability

3 Some translations

Scalar μ -calculus

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Vectorial μ -calculus

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Vectorial μ -calculus

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$$T: \begin{cases} x_1 =_{\theta_1} f_1(x_1, \dots, x_n, y_1, \dots, y_m) \\ \dots \\ x_n =_{\theta_n} f_n(x_1, \dots, x_n, y_1, \dots, y_m) \end{cases}$$

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where f_i are modal formulas and θ_i is μ or ν .

A vectorial μ -term T is by definition equivalent to a n-tuple of formulas $(Sol_1(T), \ldots, Sol_n(T))$.

Fixpoint hierarchy

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Fixpoint hierarchy

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 $\Delta_n = \Sigma_n \cap \Pi_n.$

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Kripke semantics

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Kripke semantics

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We need valued graphs (G, val), where

G = (V, R) is a graph;

val is a function from atoms and variables to $\mathcal{P}(V)$.

For every formula ϕ , $||\phi||(G, val)$ is a subset of V, defined by induction on ϕ .

Bisimulation

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Bisimulation

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S5 is the class of all equivalence relations, i.e. of reflexive, symmetric, transitive graphs.



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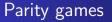
S5 is the class of all equivalence relations, i.e. of reflexive, symmetric, transitive graphs.

K4 is important in many contexts (e.g. temporal reasoning), whereas S5 is often used as an epistemic logic.



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Two players c and d (after Arnold);

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Parity games

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$$G = (V_c, V_d, E, v_0, \Omega : V_c \cup V_d \rightarrow \{1, \ldots, n\});$$

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otherwise, d wins if the largest value of Ω occurring infinitely often is even.

Preliminaries Model checking and satisfiability

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2 Model checking and satisfiability

3 Some translations

A reduction theorem

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A reduction theorem

Theorem

Given a model (G, val) and a vectorial μ -term T,

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A reduction theorem

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Given a model (G, val) and a vectorial μ -term T, there is a model (G', val') and a vectorial μ -term T', such that:

• T' is existential (i.e. box free);

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- T' is existential (i.e. box free);
- G' is of class S5;
- (G', Val') and T' are built in time polynomial in the size of (G, val) plus the size of T;
- (*G*, val) verifies sol₁(*T*) if and only if (*G'*, val') verifies sol₁(*T'*).

Corollaries

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Corollary

If there is a polytime translation from box-free vectorial μ -calculus in S5 to vectorial modal logic in S5,

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Satisfiability in S5

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Theorem

The satisfiability problem for the μ -calculus in S5 is NP-complete.

In fact, *NP*-hardness is because the μ -calculus includes propositional logic; an *NP* algorithm is given by guessing a model of a formula and then running an *NP* model checking algorithm.

On existential μ -calculus

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2 Model checking and satisfiability



The Alberucci-Facchini rank

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$$rank(A) = rank(\neg A) = 1;$$

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• $rank(\phi \land \psi) = rank(\phi \lor \psi) = max\{rank(\phi), rank(\psi)\} + 1;$

•
$$rank(\mu X.\phi(X)) = rank(\nu X.\phi(X)) =$$

 $sup\{rank(\phi^n(X)) + 1; n \in \mathbf{N}\}.$

The Alberucci-Facchini translation

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•
$$t(A) = A, t(\neg A) = \neg A;$$

•
$$t(true) = true, t(false) = false;$$

•
$$t(\langle \rangle \phi) = \langle \rangle t(\phi);$$

•
$$t([]\phi) = []t(\phi);$$

•
$$t(\phi \wedge \psi) = t(\phi) \wedge t(\psi);$$

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$$t(\nu X.\phi(X)) = t((\phi(\phi(true))^*)),$$

It is a function t given by induction on the rank:

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$$t(\phi \wedge \psi) = t(\phi) \wedge t(\psi);$$

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$$t(\phi \lor \psi) = t(\phi) \lor t(\psi)$$

•
$$t(\mu X.\phi(X)) = t((\phi(\phi(false))^*);$$

•
$$t(\nu X.\phi(X)) = t((\phi(\phi(true))^*)),$$

where $(\phi(\phi(false))^*, (\phi(\phi(true))^*$ denote the well named formulas obtained from $\phi(\phi(false)), \phi(\phi(true))$ by renaming repeated bound variables.

The complexity of the translation

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The complexity of the translation

Lemma

If ϕ is a well named formula, then the length of $t(\phi)$ is at most $2^{|\phi|}$.

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Lemma

If ϕ is a well named formula, then the length of $t(\phi)$ is at most $2^{|\phi|}$.

The exponential bound is tight.

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An alternative translation

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An alternative translation

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So, ϕ is equivalent to the finite disjunction of the characteristic formulas of the bisimulation classes of the models of ϕ . Note that this alternative translation is also (at most) exponential.

A corollary

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Conclusion

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Simplicity of S5 gives good satisfiability bounds, but no good model checking bounds.

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Are there "natural" translations from vectorial to scalar terms in S5?

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Simplicity of S5 gives good satisfiability bounds, but no good model checking bounds.

Are there "natural" translations from vectorial to scalar terms in S5?

Complexity of satisfiability of μ in K4 can be obtained by reducing to arbitrary graphs; what about better bounds?

Thank you!

D'Agostino-Lenzi On modal µ-calculus

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