On the Approximation of Mean-Payoff Games

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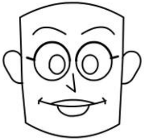
Convegno Italiano Logica Computazionale (CILC2011)

Contents

Mean-Payoff Games MPG

- 2 players:
  Maximazer B$\Box$b vs Minimazer $\triangle$lice



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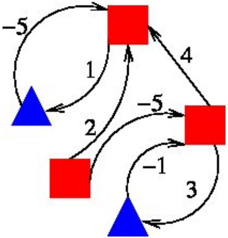

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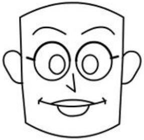

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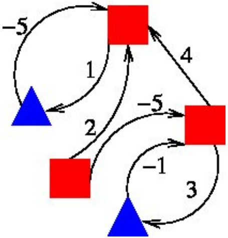

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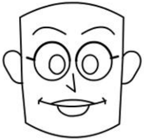

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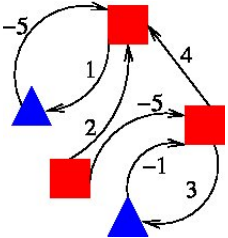

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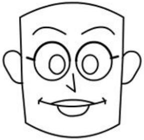

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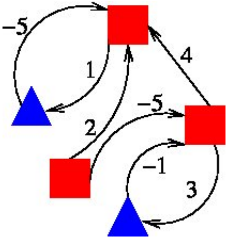

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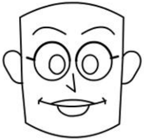


- played on a finite graph (arena)
- turn based
- infinite number of turns
- goal (for Bob): maximazing the long-run average weight

MPG in Formal Term

In a MPG $\Gamma = (V, E, w : V \to \mathbb{Z}, \langle V_\Box, V_\triangle \rangle)$:
B□b wants to maximize his payoff, i.e. the long-run average weight in a play .

Given a play $p = \{v_i\}_{i \in \mathbb{N}}$ in Γ, the payoff of B□b on p is:

$$\mathrm{MP}(v_0 v_1 \dots v_n \dots) = \liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w(v_i, v_{i+1})$$

MPG in Formal Term

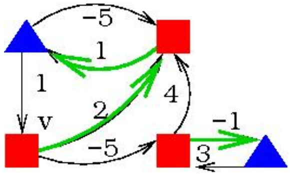
The value secured by a strategy $\sigma_\Box \colon V^* \cdot V_\Box \to V$ in vertex v is:

$$\mathrm{val}^{\sigma_\Box}(v) = \inf_{\sigma_\triangle \in \Sigma_\triangle} \mathrm{MP}(\mathrm{outcome}^\Gamma(v, \sigma_\Box, \sigma_\triangle))$$

$\sup_{\sigma_\Box \in \Sigma_\Box}(\mathrm{val}^{\sigma_\Box}(v))$ is the optimal value that B□b can secure in v

MPG in Formal Term

Example



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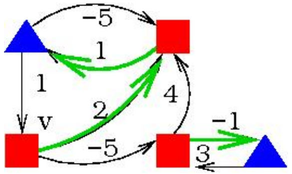
MPG in Formal Term

Example

$\mathrm{val}^{\sigma_\square}(v) = \frac{-4}{2}$



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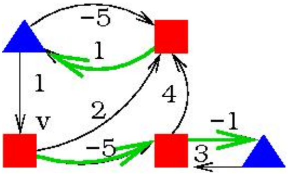

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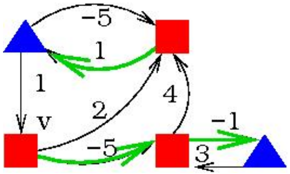
MPG in Formal Term

Example

$\sup_{\sigma_\Box \in \Sigma_\Box} (\mathrm{val}^{\sigma_\Box}(v)) = \frac{2}{2}$



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MPG are Memoryless Determined

Theorem [Ehrenfeucht&Mycielsky'79]

$$\mathrm{val}^{\Gamma}(v) = \sup_{\sigma_\square \in \Sigma_\square} \inf_{\sigma_\triangle \in \Sigma_\triangle} \mathrm{MP}(\mathrm{outcome}^{\Gamma}(v, \sigma_\square, \sigma_\triangle)) =$$
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There exist uniform memoryless strategies, $\pi_\square : V_\square \to V$ for B□b , $\pi_\triangle : V_\triangle \to V$ for △lice such that:

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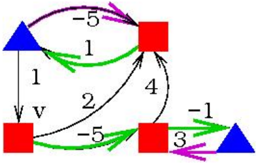

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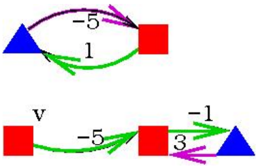

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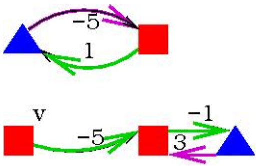

$\text{val}^\Gamma(v) = \frac{n}{d} \in \mathbb{Q} : 0 \leq d \leq |V|$ and $\frac{|n|}{d} \leq M = \max_{e\in E}\{|w(e)|\}$.

Theorem [Ehrenfeucht&Mycielsky'79]

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MPG Problems

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1. Decision Problem Given $v \in V$, $\mu \in \mathbb{Z}$, decide if B□b has a strategy $\pi_\Box$ to secure $\mathrm{val}^{\pi_\Box}(v) \geq \mu$.

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3. (Optimal) Strategy Synthesis Construct an (optimal) strategy for B□b.

MPG Problems: Why They Matter?

$\models$

Correctness Relation

Quantitative Requirements:

- limited resources
- average performance . . .

MPG Problems: Why They Matter?

$\models$

Correctness Relation

System Model?

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MPG Problems: Why They Matter?

Solved as a game: System vs Environment
Solution = Winning Strategy

$\models$

Correctness Relation

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MPG Problems: Why They Matter?

- MPG significative for theoretical and applicative aspects
  - μ-calculus model checking $\overset{\text{PTIME}}{\Longleftrightarrow}$ parity games $\overset{\text{PTIME}}{\Longrightarrow}$ MPG
  - MPG $\overset{\text{PTIME}}{\Longrightarrow}$ simple stochastic games
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  - MPG $\overset{\text{PTIME}}{\Longrightarrow}$ simple stochastic games
  - MPG $\overset{\text{PTIME}}{\Longrightarrow}$ discounted payoff games
- MPG problems have an interesting complexity status
  - MPG decision problem belongs to NP $\cap$ coNP (and even to UP $\cap$ coUP)
  - No polynomial algorithm known so far

Solving MPG Problems

Consider $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$, where $w : V \to [-M \cdots + M]$:

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- $\Theta(EV^4 M \log(\frac{E}{V}))$ algorithm for optimal strategy synthesis

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Y. Lifshits and D. Pavlov, 2006

- $\mathcal{O}(EV2^V \log(Z))$ algorithm for the decision/value problem

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Pseudopolynomial Algorithms

L. Brim, J. Chaloupka, L. Doyen, R. Gentilini and J-F. Raskin – 2010

- $\mathcal{O}(E \cdot V \cdot M)$ for the decision problem & strategy synthesis
- $\mathcal{O}(E \cdot V^2 \cdot M \cdot (\log(V) + \log(M)))$ for the value problem
- $\mathcal{O}(E \cdot V^2 \cdot M \cdot (\log(V) + \log(M)))$ algorithm for optimal strategy synthesis

Value Approximation: Basics (I)

Let $\Gamma = (V, E, w, \langle V_0, V_1 \rangle)$ be a MPG, let $v \in V$ and consider $\varepsilon \geq 0$.

Definition (MPG additive ε-value)

The value $\widetilde{val} \in \mathbb{Q}$ is said an *additive ε-value* on v if and only if:

$$|\widetilde{val} - val^{\Gamma}(v)| \leq \varepsilon$$

Definition (MPG relative ε-value)

The value $\widetilde{val} \in \mathbb{Q}$ is said an *relative ε-value* on v if and only if:

$$\frac{|\widetilde{val} - val^{\Gamma}(v)|}{|val^{\Gamma}(v)|} \leq \varepsilon$$

Value Approximation: Basics (II)

Let $\Gamma = (V, E, w, \langle V_0, V_1 \rangle)$ be a MPG:

MPG Polynomial Time Approximation Scheme (PTAS)

An additive/relative polynomial approximation scheme for Γ is an algorithm $\mathcal{A}$ such that for all $\varepsilon > 0$, $\mathcal{A}$ computes an additive/relative ε-value in time polynomial w.r.t. the size of Γ.

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MPG Fully Polynomial Time Approximation Scheme (FPTAS)

An additive/relative fully polynomial approximation scheme for Γ is an algorithm $\mathcal{A}$ such that for all $\varepsilon > 0$, $\mathcal{A}$ computes an additive/relative ε-value in time polynomial w.r.t. Γ and $\frac{1}{\varepsilon}$.

Additive Approximations – FPTAS (I)

A. Roth, M. Balcan, A. Kalai & Y. Mansour – 2010

The MPGvalue problem on graphs with rational weights in [-1,+1] admits an additive FPTAS.

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Easy approximation algorithm based on:
existing pseudopolynomial procedures + graph reweighting

Additive Approximations – FPTAS (II)

Can we efficiently approximate the value in MPG with no restriction on the weights?

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Proof (Sketch):

- By contradiction. Assume an additive FPTAS exists.
- Choose $\varepsilon = \frac{1}{2n(n-1)}$ and compute the additive ϵ-value v^ε.
- The MPG value v is the unique rational with denominator $1 \leq d \leq n$ in the interval $[v^\varepsilon - \varepsilon, v^\varepsilon + \varepsilon]$.

Additive Approximations – PTAS

Are weaker notions of approximation usefull to obtain some positive result w.r.t. the MPG value approximation problem?

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Theorem

For any constant k:
If the problem of computing an additive k-approximate MPG value can be solved in polynomial time (w.r.t. the size of the MPG), then the MPG value problem belongs to P.

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Corollary

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1. Solving the MPG value problem.
2. Determining an additive FPTAS for the MPG value problem.
3. Determining an additive PTAS for the MPG value problem.
4. Computing an additive k-approximate MPG value in polynomial time, for any constant k.

Relative Approximations (I)

Y. Boros, E. Elbassioni, M. Fouz, V. Gurvich, K. Makino i & B. Manthey – 2011

The MPG value problem on graphs with nonnegative weights admits a relative FPTAS.

Relative Approximations (II)

Can we design efficient relative approximations for the MPG value problem on graphs with no restriction on the weights?

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Theorem

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The End

Thank you!