On the satisfiability problem for a 4-level quantified syllogistic and some applications to modal logic

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Outline

- Motivation
- $4LQS^R$: a four-level quantified syllogistic
- A decision procedure for $4LQS^R$
- Applications to modal logic
- Conclusions and future work

Multi-sorted stratified syllogistics – motivation

Multi-sorted stratified syllogistics are set-theoretic languages admitting variables of different sorts (sort 0, sort 1,.. and so on)

Assignments for such variables are *based on* collections of objects (natural numbers, real numbers, possible worlds...)

- Variables of sort 0 $\rightarrow$ objects of the considered domain
- Variables of sort 1 $\rightarrow$ sets of such objects
- Variables of sort 2 $\rightarrow$ collections of sets
- ...

Multi-sorted stratified syllogistics – motivation

The study of set-theoretic languages admitting variables of different sorts is motivated by the fact that most of the formulae in the statements and proofs of theorems in many fields of mathematics and computer science involve variables of different sorts

Example:
In modal logics there are entities of different types: possible worlds, formulae, accessibility relations

Multi-sorted stratified syllogistics – motivation

Less investigated than one-sorted multi-level syllogistics

Some results:

- Two-level syllogistics, $2LS$
  (A. Ferro and E. Omodeo, 1978)
- Extension of $2LS$ with singleton and cartesian product
  (D. Cantone, V. Cutello, 1990)
- Three-level syllogistics, extended with powerset, general union, singleton
  (D. Cantone, V. Cutello 1993)

More recently, decidability of the satisfiability problem for a three-level stratified syllogistic, $3LQS^R$, admitting a restricted form of quantification over individual and set variables was proved in (D. Cantone, M. Nicolosi Asmundo 2008)

This work

We prove the decidability of the satisfiability problem for a four-level stratified syllogistic, $4LQS^R$, admitting variables of four sorts and a restricted form of quantification over variables of the first three sorts

Given a satisfiable formula φ and a model $\boldsymbol{\mathcal{M}}$

we construct a finite model $\boldsymbol{\mathcal{M}}^*$ for φ

Complexity results:
Sublanguages $(4LQS^R)^k$, $k \geq 0$ are NP-complete

Applications:
Modal logics S5 and K45 can be formalized in $(4LQS^R)^1$

Syntax of $4LQS$

- Pairing operator $\langle \cdot, \cdot \rangle$
- Predicate symbols $=$ and $\in$

(i) variables of *sort 0*: $x, y, z, \ldots$

(ii) variables of *sort 1*: $X^1, Y^1, Z^1, \ldots$

(iii) variables of *sort 2*: $X^2, Y^2, Z^2, \ldots$

(iv) variables of *sort 3*: $X^3, Y^3, Z^3, \ldots$

$4LQS$ *quantifier-free atomic formulae*

level 0**:** $x = y$, $x \in X^1$

level 1**:** $X^1 = Y^1$, $X^1 \in X^2$

level 2**:** $X^2 = Y^2$, $\langle x, y \rangle = X^2$, $\langle x, y \rangle \in X^3$, $X^2 \in X^3$

Syntax of $4LQS$

$4LQS$ *quantified atomic formulae*

level 1**:** $(\forall z_1)\ldots(\forall z_n)\varphi_0$, φ_0 propositional combination of quantifier-free atomic formulae

level 2**:** $(\forall Z_1^1)\ldots(\forall Z_m^1)\varphi_1$, φ_1 propositional combination of quantifier-free atomic formulae and of quantified atomic formulae of level 1

level 3**:** $(\forall Z_1^2)\ldots(\forall Z_p^2)\varphi_2$, φ_2 propositional combination of quantifier-free atomic formulae and of quantified atomic formulae of levels 1 and 2

$4LQS$-**Formulae**

Propositional combinations of quantifier-free atomic formulae of levels $0, 1$, 2, and of quantified atomic formulae of levels 1, 2, 3

Semantics of $4LQS$

A $4LQS$-interpretation is a pair $\mathcal{M} = (D, M)$, where

- $Mx \in D$
- $MX^1 \in \mathrm{pow}(D)$
- $MX^2 \in \mathrm{pow}(\mathrm{pow}(D))$
- $MX^3 \in \mathrm{pow}(\mathrm{pow}(\mathrm{pow}(D)))$

We put $M\langle x, y\rangle = \{\{Mx\}, \{Mx, My\}\}$

Semantics of $4LQS$

Formulae are interpreted as usual. In particular

$\boldsymbol{\mathcal{M}} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$ iff $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \models \varphi_0$, for all $u_1, \ldots, u_n \in D$

$\boldsymbol{\mathcal{M}} \models (\forall Z_1^1) \ldots (\forall Z_m^1)\varphi_1$ iff $\boldsymbol{\mathcal{M}}[Z_1^1/U_1^1, \ldots, Z_m^1/U_m^1] \models \varphi_1$, for all $U_1^1, \ldots, U_m^1 \in \mathrm{pow}(D)$

$\boldsymbol{\mathcal{M}} \models (\forall Z_1^2) \ldots (\forall Z_p^2)\varphi_2$ iff $\boldsymbol{\mathcal{M}}[Z_1^2/U_1^2, \ldots, Z_p^2/U_p^2] \models \varphi_2$, for all $U_1^2, \ldots, U_p^2 \in \mathrm{pow}(\mathrm{pow}(D))$

Examples

- $(\forall z)(z \in X^1 \leftrightarrow \varphi(z))$
- $(\forall X^{i-1})(X^{i-1} \in X^i \leftrightarrow \varphi(X^{i-1}))$, for $i \in \{2,3\}$

For instance

$$(\forall Z^2)(Z^2 \in Y^3 \leftrightarrow$$

$$(\forall Z^1)(Z^1 \in Z^2 \leftrightarrow$$

$$\neg(\forall z_1)(\forall z_2)(\forall z_3)\neg(z_1 \in Z^1 \wedge z_2 \in Z^1 \wedge z_3 \in Z^1 \wedge$$

$$\neg(z_1 = z_2) \wedge \neg(z_1 = z_3) \wedge \neg(z_2 = z_3))))$$

Characterizing $4LQS^R$

$4LQS^R$ is the subcollection of the formulae of $4LQS$ such that

Restriction I

Nestings of quantifiers over variables of sort 0 into quantifiers over variables of sort 1 are allowed if the former are linked to the corresponding variables of sort 1

$[\neg \varphi_0 \rightarrow \wedge_{i=1}^{n} \vee_{j=1}^{m} z_i \in Z_j^1]$

Example

$$(\forall Z^1)(Z^1 \in X^2 \leftrightarrow (\forall z)((z \in Z^1 \rightarrow z \in X^1))$$

If $\mathcal{M} \models \neg(z \in Z^1 \rightarrow z \in X^1)$ then $\mathcal{M} \models z \in Z^1$

Characterizing $4LQS^R$

Restriction II

Every quantified atomic formula of level 3 is

- either of type $(\forall Z_1^2), \ldots, (\forall Z_p^2)\varphi_2$, where φ_2 is a propositional combination of quantifier-free atomic formulae
- or of type $(\forall Z^2)(Z^2 \in X^3 \leftrightarrow \neg(\forall z_1)(\forall z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$

Examples

- $(\forall Z^2)(Z^2 \in X^3 \leftrightarrow (Z^2 \in X_1^3 \wedge Z^2 \in X_2^3))$ (intersection)
- $(\forall Z^2)(Z^2 \in X^3 \leftrightarrow (Z^2 \in X_1^3 \wedge \neg(Z^2 \in X_2^3)))$ (set difference)

A decision procedure for the satisfiability problem for $4LQS^R$

Given a satisfiable $4LQS^R$-formula φ and a model $\boldsymbol{\mathcal{M}}$ for it, we construct a finite $4LQS^R$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$, where $D^* \subseteq D$ is finite

1. Normalized $4LQS^R$-conjunctions
2. Construction of D^*
3. Definition of M^*

Normalized $4LQS^R$-conjunctions

Let φ be a $4LQS^R$-formula

construct φ_{DNF} and consider only one of its disjuncts

Negative quantified conjuncts occurring in it are eliminated

Each

$\neg(\forall z_1)\ldots(\forall z_n)\varphi_0$ is replaced by $\neg(\varphi_0)^{z_1,\ldots,z_n}_{z'_1,\ldots,z'_n}$

negative quantified literals of levels 2 and 3 are dealt with in an analogous way

Example

$$\varphi = (\forall Z^1)(Z^1 \in X_1^2 \to (\forall z)(z \in Z^1 \to z \in X^1))$$
$$\wedge \neg(X_1^2 = X_2^2) \wedge \neg(\forall z_1)(\forall z_2)\neg((\langle z_1, z_2 \rangle = X_2^2 \wedge X_2^2 \in X^3)$$

$$\Downarrow$$

$$\varphi_N = (\forall Z^1)(Z^1 \in X_1^2 \to (\forall z)(z \in Z^1 \to z \in X^1))$$
$$\wedge \neg(X_1^2 = X_2^2) \wedge \langle x_1, x_2 \rangle = X_2^2 \wedge X_2^2 \in X^3$$

A model for φ_N

$$\varphi_N = (\forall Z^1)(Z^1 \in X_1^2 \to (\forall z)(z \in Z^1 \to z \in X^1))$$
$$\wedge \neg(X_1^2 = X_2^2) \wedge \langle x_1, x_2 \rangle = X_2^2 \wedge X_2^2 \in X^3$$

$\boldsymbol{\mathcal{M}} = (D, M)$ where

$D = \mathbb{N}$

Mx_1, $Mx_2 \in \mathbb{N}$

$MX^1 = \mathbb{N}$

MX_1^2 the collection of subsets of $\mathbb{N}$ with at least three elements

$MX_2^2 = \{\{Mx_1\}, \{Mx_1, Mx_2\}\}$

$MX_2^2 \in MX^3$

Construction of the finite domain D^*

$\mathcal{V}'_0$, $\mathcal{V}'_1$, $\mathcal{V}'_2$ the collections of variables of sorts 0, 1, 2 free in φ_N

Step 1: find distinguishers and witnesses of cardinality at level 2

- $\mathcal{F}_1$ 'distinguishes' $S = \{MX^2 : X^2 \in \mathcal{V}'_2\}$, and $|\mathcal{F}_1| \leq |S| - 1$
- $\mathcal{F}_2$ satisfies $|MX^2 \cap \mathcal{F}_2| \geq \min(3, |MX^2|)$, and $|\mathcal{F}_2| \leq 3 \cdot |\mathcal{V}'_2|$

Example

- $S = \{MX^2_1, MX^2_2\}$
- $\mathcal{F}_1 = \{F\}$ with $F \subseteq \mathbb{N}$, $F \in MX^2_1$, $F \notin MX^2_2$
- $\mathcal{F}_2 = \{F_1, F_2, F_3, F'_1, F'_2\}$,
  where $F_1, F_2, F_3 \in MX^2_1$ and $F'_1, F'_2 \in MX^2_2$

Construction of the finite domain D^*

Step 2: associate variables to distinguishers and witnesses of cardinality

- $\{F_1, \ldots, F_k\} = (\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{MX^1 : X^1 \in \mathcal{V}_1'\}$
- $\mathcal{V}_1^F = \{X_1^1, \ldots, X_k^1\} \subseteq \mathcal{V}_1$ distinct from the variables in φ_N

Example

Assume $MX^1 \notin \mathcal{F}_1 \cup \mathcal{F}_2$

- $\mathcal{V}_1^F = \{Y, Y_1, Y_2, Y_3, Y_1', Y_2'\}$
- $MY = F$
- $MY_i = F_i$
- $MY_j' = F_j'$

Construction of the finite domain D^*

Step 3: find distinguishers and witnesses of cardinality at level 1

- Δ_1 distinguishes $T = \{MX^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}$
  - ▶ $|\Delta_1| \leq |T| - 1$
- Δ_2 satisfies $|J \cap \Delta_2| \geq \min(3, |J|)$
  - ▶ $|\Delta_2| \leq 3 \cdot |\mathcal{V}_1' \cup \mathcal{V}_1^F|$

- $D^* = \{Mx : x \in \mathcal{V}_0'\} \cup \Delta_1 \cup \Delta_2$

Example

- $\Delta_1 = \{u_1, \ldots, u_k\}$, $k \leq 6$
- $\Delta_2 = \{v_1, \ldots, v_l\}$, $l \leq 3 \cdot 5 + 1 + 2$

Construction of the finite domain D^*

Step 4: complete D^* to preserve satisfiability of level 1 literals

Let Φ be the set of formulae $(\forall z_1)\ldots(\forall z_n)\varphi_0$ in conjuncts of type $(\forall Z_1^1)\ldots(\forall Z_m^1)\varphi_1$

- for every $\varphi \in \Phi$ and for each $(X_{i_1}^1, \ldots, X_{i_m}^1)$

$$\text{if } \mathcal{M}\varphi^{Z_1^1, \ldots, Z_m^1}_{X_{i_1}^1, \ldots, X_{i_m}^1} = \mathbf{false} \text{ then}$$

$$D^* := D^* \cup \{u_1, \ldots, u_n\}, \quad u_1, \ldots, u_n \text{ of } D$$

$$\mathcal{M}[z_1/u_1, \ldots, z_n/u_n]\varphi_0{}^{Z_1^1, \ldots, Z_m^1}_{X_{i_1}^1, \ldots, X_{i_m}^1} = \mathbf{false}$$

Definition of the interpretation M^*

$M^* x = Mx$, if $Mx \in D^*$

$M^* X^1 = MX^1 \cap D^*$

$$M^* X^2 = ((MX^2 \cap \mathrm{pow}(D^*)) \setminus \{M^* X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)\}) \cup \{M^* X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1),\ MX^1 \in MX^2\}$$

$$M^* X^3 = ((MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^* X^2 : X^2 \in \mathcal{V}'_2\}) \cup \{M^* X^2 : X^2 \in \mathcal{V}'_2,\ MX^2 \in MX^3\}$$

Correctness of the procedure

Let $\mathcal{M}$ be a $4LQS^R$-interpretation satisfying a normalized $4LQS^R$-conjunction φ_N. Further, let $\mathcal{M}^* = (D^*, M^*)$ be a $4LQS^R$-interpretation defined as shown in the procedure above. Then $\mathcal{M}^* \models \varphi_N$.

The correctness proof is carried out by means of some technical lemmas showing that

- for every quantifier-free literal ψ of φ_N, $\mathcal{M} \models \psi$ iff $\mathcal{M}^* \models \psi$
- for every quantified literal χ of φ_N, if $\mathcal{M} \models \chi$ then $\mathcal{M}^* \models \chi$

Complexity results

$(4LQS^R)^k$ are sublanguages of $4LQS^R$ in which the quantifier prefixes of quantified atoms of level 2 have length not exceeding k

Lemma The satisfiability problem for $(4LQS^R)^k$ is NP-complete, for any $k \geq 0$

- NP-hardness is proved by a reduction to the satisfiability problem of propositional logic
- The problem is in NP:
  - $|D^*|$ is polynomial in the size of φ_N (polynomial in the size of φ)
  - $\mathcal{M}^* \models \varphi_N$ can be verified in polynomial time

Expressibility of $4LQS^R$

- Set theoretic constructs expressed by $3LQS^R$
- Binary relations
- Conditions on binary relations which characterize accessibility relations of modal logics
- Boolean operations over relations and the inverse of a given binary relation
- The modal logics S5 and K45
- ...

$4LQS^R$ formalization of conditions of accessibility relations

Binary relation	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2))$
Reflexive	$(\forall z_1)(\langle z_1, z_1\rangle \in R^3)$
Symmetric	$(\forall z_1, z_2)(\langle z_1, z_2\rangle \in R^3 \rightarrow \langle z_2, z_1\rangle \in R^3)$
Transitive	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_3\rangle \in R^3) \rightarrow \langle z_1, z_3\rangle \in R^3)$
Euclidean	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_1, z_3\rangle \in R^3) \rightarrow \langle z_2, z_3\rangle \in R^3)$
Weakly-connected	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_1, z_3\rangle \in R^3)$ $\rightarrow (\langle z_2, z_3\rangle \in R^3 \vee z_2 = z_3 \vee \langle z_3, z_2\rangle \in R^3))$
Irreflexive	$(\forall z_1)\neg(\langle z_1, z_1\rangle \in R^3)$
Intransitive	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_3\rangle \in R^3) \rightarrow \neg\langle z_1, z_3\rangle \in R^3)$
Antisymmetric	$(\forall z_1, z_2)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_1\rangle \in R^3) \rightarrow (z_1 = z_2))$
Asymmetric	$(\forall z_1, z_2)(\langle z_1, z_2\rangle \in R^3 \rightarrow \neg(\langle z_2, z_1\rangle \in R^3))$

$4LQS^R$ formalization of Boolean operations over relations

Intersection	$R^3 = R^3_1 \cap R^3_2$	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow (Z^2 \in R^3_1 \wedge Z^2 \in R^3_2))$
Union	$R^3 = R^3_1 \cup R^3_2$	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow (Z^2 \in R^3_1 \vee Z^2 \in R^3_2))$
Complement	$R^3_1 = \overline{R^3_2}$	$(\forall Z^2)(Z^2 \in R^3_1 \leftrightarrow \neg(Z^2 \in \overline{R^3_2}))$
Set difference	$R^3 = R^3_1 \setminus R^3_2$	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow (Z^2 \in R^3_1 \wedge \neg(Z^2 \in R^3_2)))$
Set inclusion	$R^3_1 \subseteq R^3_2$	$(\forall Z^2)(Z^2 \in R^3_1 \rightarrow Z^2 \in R^3_2)$

The logic S5

Modal logic S5 can be obtained from the logic K by adding to

K : $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$,

the axioms

T : $\Box p \to p$ (reflexive accessibility relation)
$\mathbf{5}$: $\Diamond p \to \Box\Diamond p$ (euclidean accessibility relation)

Semantics of the modal operators

- $K, w \models \Box\varphi$ iff $K, v \models \varphi$, for every $v \in W$,
- $K, w \models \Diamond\varphi$ iff $K, v \models \varphi$, for some $v \in W$.

Translation of a formula of S5 in $4LQS^R$

- worlds $\rightarrow$ variables of sort 0
- formulae $\rightarrow$ variables of sort 1
- relations $\rightarrow$ variables of sort 3

$$
\begin{aligned}
\tau_{\mathsf{S5}}(\Box p) &= \tau^2_{\mathsf{S5}}(\Box p) \\
&= (\forall z)(z \in X^1_p) \rightarrow (\forall z)(z \in X^1_{\Box p}) \\
&\quad \wedge \neg(\forall z)(z \in X^1_p) \rightarrow (\forall z)\neg(z \in X^1_{\Box p})
\end{aligned}
$$

Lemma

For every formula φ of the logic S5, φ is satisfiable in a model $K = \langle W, R, h \rangle$ iff there is a $4LQS^R$-interpretation satisfying $x \in X_\varphi$.

The logic K45

Modal logic K45 can be obtained from the logic K by adding to

K : $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$,

the axioms

$\mathbf{4}$: $\Box p \to \Box\Box p$ (transitive accessibility relation)
$\mathbf{5}$: $\Diamond p \to \Box\Diamond p$ (euclidean accessibility relation)

Semantics of the modal operators

- $K \models \Box\varphi$ iff $K, v \models \varphi$, for every $v \in W$ s.t. there is a $w' \in W$ with $(w', v) \in R$,
- $K \models \Diamond\varphi$ iff $K, v \models \varphi$, for some $v \in W$ s.t. there is a $w' \in W$ with $(w', v) \in R$.

Translation of a formula of K45 in $4LQS^R$

$\tau_{\mathsf{K45}}(\Box p)$

$= \tau^2_{\mathsf{K45}}(\Box p)$

$= (\forall z_1)((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \to z_1 \in X^1_p)$

$\to (\forall z)(z \in X^1_{\Box p})$

$\wedge\neg(\forall z_1)\neg((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \wedge \neg(z_1 \in X^1_p))$

$\to (\forall z)\neg(z \in X^1_{\Box p})$

Lemma

Given a formula φ of τ_{K45}, φ is satisfiable in $K = \langle W, R, h\rangle$ iff there is a $4LQS^R$- interpretation satisfying $x \in X_\varphi$

Conclusions

We have presented a decidability result for the satisfiability problem for a 4-level quantified syllogistic ($4LQS^R$)

$4LQS^R$ expresses in a natural way

- pairs
- (binary) relations
- Boolean operations over relations
- properties of binary relations

The modal logics S5 and K45 can be embedded into $4LQS^R$

Future work

- Exploring the problem of expressing:
  - Set theoretic general union
  - Composition of binary relations
  - The modal logic K
- Characterizing the conditions a modal logic has to fulfil in order to be embedded into $4LQS^R$
- Finding classes of
  - Modal formulae with bounded modal nesting
  - Multi-modal logics

  that can be embedded into $4LQS^R$