On the satisfiability problem for a 4-level quantified syllogistic and some applications to modal logic

Domenico Cantone and Marianna Nicolosi Asmundo

Dipartimento di Matematica e Informatica, Università di Catania
Viale A. Doria 6, I-95125 Catania, Italy
e-mail: cantone@dmi.unict.it, nicolosi@dmi.unict.it

Abstract. We introduce a fragment of multi-sorted stratified syllogistic, called $4LQS^R$, admitting variables of four sorts and a restricted form of quantification, and prove that it has a solvable satisfiability problem by showing that it enjoys a small model property. Then, we consider the sublanguage $(4LQS^R)^k$ of $4LQS^R$, where the length of quantifier prefixes (over variables of sort 1) is bounded by $k \geq 0$, and prove that its satisfiability problem is NP-complete. Finally we show that modal logics S5 and K45 can be expressed in $(4LQS^R)^1$.

1 Introduction

Most of the decidability results in computable set theory concern one-sorted multi-level syllogistics, namely collections of formulae admitting variables of one sort only, which range over the von Neumann universe of sets (see [6, 8] for a thorough account of the state-of-art until 2001). Only a few stratified syllogistics, where variables of several sorts are allowed, have been investigated, despite the fact that in many fields of computer science and mathematics often one has to deal with multi-sorted languages. For instance, in modal logics, one has to consider entities of different types, namely worlds, formulae, and accessibility relations.

In [10] an efficient decision procedure was presented for the satisfiability of the Two-Level Syllogistic language ($2LS$). $2LS$ has variables of two sorts and admits propositional connectives together with the basic set-theoretic operators $\cup, \cap, \setminus$, and the predicates $=, \in$, and $\subseteq$. Then, in [2], it was shown that the extension of $2LS$ with the singleton operator and the Cartesian product operator is decidable. Tarski's and Presburger's arithmetics extended with sets have been analyzed in [4]. Subsequently, in [3], a three-sorted language $3LSSPU$ (Three-Level Syllogistic with Singleton, Powerset and general Union) has been proved decidable. Recently, in [7], it was shown that the Three-Level Quantified Syllogistic with Restricted quantifiers language ($3LQS^R$) is decidable. $3LQS^R$ admits variables of three sorts and a restricted form of quantification. Its vocabulary contains only the predicate symbols $=$ and $\in$. In spite of that, $3LQS^R$ allows to express several constructs of set theory. Among them, the most comprehensive one is the set former, which in turn enables one to express other operators like the powerset operator, the singleton operator, and so on.

In this paper we present a decidability result for the satisfiability problem of the set-theoretic language $4LQS^R$ (Four-Level Quantified Syllogistic with Restricted quantifiers). $4LQS^R$ is an extension of $3LQS^R$ which admits variables of four sorts and a restricted form of quantification over variables of the first three sorts. Its vocabulary contains the pairing operator $\langle \cdot, \cdot \rangle$ and the predicate symbols $=$ and $\in$.

We will prove that $4LQS^R$ enjoys a small model property by showing how one can extract, out of a given model satisfying a $4LQS^R$-formula ψ, another model of ψ but of bounded finite cardinality. The construction of the finite model extends the decision algorithm described in [7]. Concerning complexity issues, we will show that the satisfiability problem for each of the sublanguages $(4LQS^R)^k$ of $4LQS^R$, whose formulae are restricted to have quantifier prefixes over variables of sort 1 of length at most $k \geq 0$, is NP-complete.

Clearly, $4LQS^R$ can express all the set-theoretical constructs which are already expressible by $3LQS^R$. In addition, in $4LQS^R$ one can plainly formalize several properties of binary relations also needed to define accessibility relations of well-known modal logics. $4LQS^R$ can also express Boolean operations over relations and the inverse operation over binary relations. Finally, we will show that the modal logics S5 and K45 can be easily formalized in the $(4LQS^R)^1$ language. Since the satisfiability problems for S5 and K45 are NP-complete, in terms of computational complexity the algorithm we present here can be considered optimal for both logics.

2 The language $4LQS^R$

Before defining the language $4LQS^R$, in Section 2.1 we present the syntax and the semantics of a more general four-level quantified fragment, denoted $4LQS$. Then, in Section 2.2, we introduce some restrictions over the quantified formulae of $4LQS$ which characterize $4LQS^R$-formulae.

2.1 The more general language $4LQS$

Syntax of $4LQS$. The four-level quantified language $4LQS$ involves four collections $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ of variables.

(i) $\mathcal{V}_0$ contains variables of *sort 0*, denoted by $x, y, z, \ldots$;
(ii) $\mathcal{V}_1$ contains variables of *sort 1*, denoted by $X^1, Y^1, Z^1, \ldots$;
(iii) $\mathcal{V}_2$ contains variables of *sort 2*, denoted by $X^2, Y^2, Z^2, \ldots$;
(iv) $\mathcal{V}_3$ contains variables of *sort 3*, denoted by $X^3, Y^3, Z^3, \ldots$.

$4LQS$ *quantifier-free atomic formulae* are classified as:

level 0: $x = y$, $x \in X^1$, for $x, y \in \mathcal{V}_0, X^1 \in \mathcal{V}_1$;
level 1: $X^1 = Y^1$, $X^1 \in X^2$, for $X^1, Y^1 \in \mathcal{V}_1, X^2 \in \mathcal{V}_2$;
level 2: $X^2 = Y^2$, $\langle x, y \rangle = X^2$, $\langle x, y \rangle \in X^3$, $X^2 \in X^3$, for $X^2, Y^2 \in \mathcal{V}_2$, $x, y \in \mathcal{V}_0$, $X^3 \in \mathcal{V}_3$.

$4LQS$ *quantified atomic formulae* are classified as:

level 1: $(\forall z_1) \ldots (\forall z_n)\varphi_0$, with φ_0 any propositional combination of quantifier-free atomic formulae, and $z_1, \ldots, z_n$ variables of sort 0;

level 2: $(\forall Z_1^1)\dots(\forall Z_m^1)\varphi_1$, with φ_1 any propositional combination of quantifier-free atomic formulae and of quantified atomic formulae of level 1, and $Z_1^1,\dots,Z_m^1$ variables of sort 1;

level 3: $(\forall Z_1^2)\dots(\forall Z_p^2)\varphi_2$, with φ_2 any propositional combination of quantifier-free atomic formulae and of quantified atomic formulae of levels 1 and 2, and $Z_1^2,\dots,Z_p^2$ variables of sort 2.

Finally, the formulae of $4LQS$ are all the propositional combinations of quantifier-free atomic formulae of levels $0, 1$, 2, and of quantified atomic formulae of levels 1, 2, 3.

Semantics of $4LQS$. A $4LQS$*-interpretation* is a pair $\boldsymbol{\mathcal{M}} = (D, M)$, where D is any *nonempty* collection of objects, called the *domain* or *universe* of $\boldsymbol{\mathcal{M}}$, and M is an assignment over the variables of $4LQS$ such that

– $Mx \in D$, for each $x \in \mathcal{V}_0$;
– $MX^1 \in \mathrm{pow}(D)$, for each $X^1 \in \mathcal{V}_1$;
– $MX^2 \in \mathrm{pow}(\mathrm{pow}(D))$, for all $X^2 \in \mathcal{V}_2$;
– $MX^3 \in \mathrm{pow}(\mathrm{pow}(\mathrm{pow}(D)))$, for all $X^3 \in \mathcal{V}_3$.[1]

Moreover we put $M\langle x, y\rangle = \{\{Mx\}, \{Mx, My\}\}$. Let

- $\boldsymbol{\mathcal{M}} = (D, M)$ be a $4LQS$-interpretation,
- $x_1, \dots, x_n \in \mathcal{V}_0$,
- $X_1^1, \dots, X_m^1 \in \mathcal{V}_1$,
- $X_1^2, \dots, X_p^2 \in \mathcal{V}_2$,
- $u_1, \dots, u_n \in D$,
- $U_1^1, \dots, U_m^1 \in \mathrm{pow}(D)$,
- $U_1^2, \dots, U_p^2 \in \mathrm{pow}(\mathrm{pow}(D))$.

By $\boldsymbol{\mathcal{M}}[x_1/u_1, \dots, x_n/u_n, X_1^1/U_1^1, \dots, X_m^1/U_m^1, X_1^2/U_1^2, \dots, X_p^2/U_p^2]$, we denote the interpretation $\boldsymbol{\mathcal{M}}' = (D, M')$ such that $M'x_i = u_i$, for $i = 1, \dots, n$, $M'X_j^1 = U_j^1$, for $j = 1, \dots, m$, $M'X_k^2 = U_k^2$, for $k = 1, \dots, p$, and which otherwise coincides with $\boldsymbol{\mathcal{M}}$ on all remaining variables. Throughout the paper we use the abbreviations: $\mathcal{M}^z$ for $\boldsymbol{\mathcal{M}}[z_1/u_1, \dots, z_n/u_n]$, $\mathcal{M}^{Z^1}$ for $\boldsymbol{\mathcal{M}}[Z_1^1/U_1^1, \dots, Z_m^1/U_m^1]$, and $\mathcal{M}^{Z^2}$ for $\boldsymbol{\mathcal{M}}[Z_1^2/U_1^2, \dots, Z_p^2/U_p^2]$.

Let φ be a $4LQS$-formula and let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $4LQS$-interpretation. The notion of *satisfiability* of φ by $\boldsymbol{\mathcal{M}}$ (denoted by $\boldsymbol{\mathcal{M}} \models \varphi$) is defined inductively over the structure of the formula. Quantifier-free atomic formulae are interpreted in the standard way according to the usual meaning of the predicates '$=$' and '$\in$', and quantified atomic formulae are evaluated as follows:

1. $\boldsymbol{\mathcal{M}} \models (\forall z_1)\dots(\forall z_n)\varphi_0$ iff $\boldsymbol{\mathcal{M}}[z_1/u_1, \dots, z_n/u_n] \models \varphi_0$, for all $u_1, \dots, u_n \in D$;
2. $\boldsymbol{\mathcal{M}} \models (\forall Z_1^1)\dots(\forall Z_m^1)\varphi_1$ iff $\boldsymbol{\mathcal{M}}[Z_1^1/U_1^1, \dots, Z_m^1/U_m^1] \models \varphi_1$, for all $U_1^1, \dots, U_m^1 \in \mathrm{pow}(D)$;

[1] We recall that, for any set s, $\mathrm{pow}(s)$ denotes the *powerset* of s, i.e., the collection of all subsets of s.

3. $\boldsymbol{\mathcal{M}} \models (\forall Z_1^2) \ldots (\forall Z_p^2)\varphi_2$ iff $\boldsymbol{\mathcal{M}}[Z_1^2/U_1^2, \ldots, Z_p^2/U_p^2] \models \varphi_2$, for all $U_1^2, \ldots, U_p^2 \in \mathrm{pow}(\mathrm{pow}(D))$.

Finally, evaluation of compound formulae plainly follows the standard rules of propositional logic. Let ψ be a $4LQS$-formula, if $\boldsymbol{\mathcal{M}} \models \psi$, i.e. $\boldsymbol{\mathcal{M}}$ *satisfies* ψ, then $\boldsymbol{\mathcal{M}}$ is said to be a $4LQS$-*model* for ψ. A $4LQS$-formula is said to be *satisfiable* if it has a $4LQS$-model. A $4LQS$-formula is *valid* if it is satisfied by all $4LQS$-interpretations.

2.2 Characterizing $4LQS^R$

$4LQS^R$ is the subcollection of the formulae ψ of $4LQS$ for which the following restrictions hold.

I. For *every* atomic formula $(\forall Z_1^1), \ldots, (\forall Z_m^1)\varphi_1$ of level 2 occurring in ψ and *every* level 1 atomic formula of the form $(\forall z_1) \ldots (\forall z_n)\varphi_0$ occurring in φ_1, φ_0 is a propositional combination of level 0 atoms and the condition

$$\neg\varphi_0 \rightarrow \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} z_i \in Z_j^1 \tag{1}$$

is a valid $4LQS$-formula (in this case we say that the atom $(\forall z_1) \ldots (\forall z_n)\varphi_0$ is *linked* to the variables $Z_1^1, \ldots, Z_m^1$).

II. *Every* atomic formula of level 3 occurring in ψ is either of type $(\forall Z_1^2), \ldots, (\forall Z_p^2)\varphi_2$, where φ_2 is a propositional combination of quantifier-free atomic formulae, or of type $(\forall Z^2)(Z^2 \in X^3 \leftrightarrow \neg(\forall z_1)(\forall z_2)\neg(\langle z_1, z_2\rangle = Z^2)$.

Restriction (I) is similar to the one described in [7]. In particular, following [7], we recall that condition (1) guarantees that if a given interpretation assigns to $z_1, \ldots, z_n$ elements of the domain that make φ_0 false, then such elements must be contained in at least one of the sets assigned to $Z_1^1, \ldots, Z_m^1$. This fact is needed in the proof of statement (ii) of Lemma 5 to make sure that satisfiability is preserved in a suitable finite submodel (details, however, are not reported here and can be found in [7]).

Through several examples, in [7] it is argued that condition (1) is not particularly restrictive. Indeed, to establish whether a given $4LQS$-formula is a $4LQS^R$-formula, since condition (1) is a $2LS$-formula, its validity can be checked using the decision procedure in [10], as $4LQS$ is a conservative extension of $2LS$. In addition, in many cases of interest, condition (1) is just an instance of the simple propositional tautology $\neg(A \rightarrow B) \rightarrow A$, and thus its validity can be established just by inspection.

Restriction (II) has been introduced to be able to express binary relations and several operations on relations keeping low, at the same time, the complexity of the decision procedure of Section 3.2.

Finally, we observe that though the semantics of $4LQS^R$ plainly coincides with the one given above for $4LQS$-formulae, in what follows we prefer to refer to $4LQS$-interpretations of $4LQS^R$-formulae as $4LQS^R$-*interpretations*.

3 The satisfiability problem for $4LQS^R$-formulae

We will solve the satisfiability problem for $4LQS^R$, i.e. the problem of establishing for any given formula of $4LQS^R$ whether it is satisfiable or not, as follows:

(i) firstly, we will show how to reduce effectively the satisfiability problem for $4LQS^R$-formulae to the satisfiability problem for *normalized* $4LQS^R$*-conjunctions* (these will be defined below);

(ii) secondly, we will prove that the collection of normalized $4LQS^R$-conjunctions enjoys a small model property.

From (i) and (ii), the solvability of the satisfiability problem for $4LQS^R$ follows immediately. Additionally, by further elaborating on point (i), it could easily be shown that indeed the whole collection of $4LQS^R$-formulae enjoys a small model property.

3.1 Normalized $4LQS^R$-conjunctions

Let ψ be a formula of $4LQS^R$ and let ψ_{DNF} be a disjunctive normal form of ψ. Then ψ is satisfiable if and only if at least one of the disjuncts of ψ_{DNF} is satisfiable. We recall that the disjuncts of ψ_{DNF} are conjunctions of literals, namely atomic formulae or their negation. In view of the previous observations, without loss of generality, we can suppose that our formula ψ is a conjunction of level $0, 1, 2$ quantifier-free literals and of level $1, 2, 3$ quantified literals. In addition, we can also assume that no variable occurs both bound and free in ψ and that distinct occurrences of quantifiers bind distinct variables.

For decidability purposes, negative quantified conjuncts occurring in ψ can be eliminated as follows. Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a model for ψ, and let $\neg(\forall z_1)\ldots(\forall z_n)\varphi_0$ be a negative quantified literal of level 1 occurring in ψ. Since $\boldsymbol{\mathcal{M}} \models \neg(\forall z_1)\ldots(\forall z_n)\varphi_0$ if and only if $\mathcal{M}[z_1/u_1, \ldots, z_n/u_n] \models \neg\varphi_0$, for some $u_1, \ldots, u_n \in D$, we can replace $\neg(\forall z_1)\ldots(\forall z_n)\varphi_0$ in ψ by $\neg(\varphi_0)^{z_1,\ldots,z_n}_{z'_1,\ldots,z'_n}$, where $z'_1, \ldots, z'_n$ are newly introduced variables of sort 0. Negative quantified literals of levels 2 and 3 can be dealt with much in the same way and hence, we can further assume that ψ is a conjunction of literals of the following types:

(1) quantifier-free literals of any level;

(2) quantified atomic formulae of level 1;

(3) quantified atomic formulae of levels 2 and 3 satisfying the restrictions given in Section 2.2.

We call these formulae *normalized* $4LQS^R$*-conjunctions*.

3.2 A small model property for normalized $4LQS^R$-conjunctions

In view of the above reductions, we can limit ourselves to consider the satisfiability problem for normalized $4LQS^R$-conjunctions only.

Thus, let ψ be a normalized $4LQS^R$-conjunction and assume that $\boldsymbol{\mathcal{M}} = (D, M)$ is a model for ψ.

We show how to construct, out of $\boldsymbol{\mathcal{M}}$, a finite $4LQS^R$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ which is a model of ψ and such that the size of D^* depends solely on the size of ψ. We will proceed as follows. First we outline a procedure for the construction of a suitable nonempty finite universe $D^* \subseteq D$. Then we show how to relativize $\boldsymbol{\mathcal{M}}$ to D^* according to Definition 1 below, thus defining a finite $4LQS^R$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$. Finally, we prove that $\boldsymbol{\mathcal{M}}^*$ satisfies ψ.

Construction of the universe D^*. Let us denote by $\mathcal{V}_0'$, $\mathcal{V}_1'$, and $\mathcal{V}_2'$ the collections of variables of sort 0, 1, and 2 occurring free in ψ, respectively. Then we construct D^* according to the following steps:

Step 1: Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where
- $\mathcal{F}_1$ 'distinguishes' the set $S = \{MX^2 : X^2 \in \mathcal{V}_2'\}$, in the sense that $K \cap \mathcal{F}_1 \neq K' \cap \mathcal{F}_1$ for every distinct $K, K' \in S$. Such a set $\mathcal{F}_1$ can be constructed by the procedure *Distinguish* described in [5]. As shown in [5], we can also assume that $|\mathcal{F}_1| \leq |S| - 1$.
- $\mathcal{F}_2$ satisfies $|MX^2 \cap \mathcal{F}_2| \geq \min(3, |MX^2|)$, for every $X^2 \in \mathcal{V}_2'$. Plainly, we can also assume that $|\mathcal{F}_2| \leq 3 \cdot |\mathcal{V}_2'|$.

Step 2: Let $\{F_1, \ldots, F_k\} = \mathcal{F} \setminus \{MX^1 : X^1 \in \mathcal{V}_1'\}$ and let $\mathcal{V}_1^F = \{X_1^1, \ldots, X_k^1\} \subseteq \mathcal{V}_1$ be such that $\mathcal{V}_1^F \cap \mathcal{V}_1' = \emptyset$ and $\mathcal{V}_1^F \cap \mathcal{V}_1^B = \emptyset$, where $\mathcal{V}_1^B$ is the collection of bound variables in ψ. Let $\overline{\boldsymbol{\mathcal{M}}}$ be the interpretation $\boldsymbol{\mathcal{M}}[X_1^1/F_1, \ldots, X_k^1/F_k]$. Since the variables in $\mathcal{V}_1^F$ do not occur in ψ (neither free nor bound), their evaluation is immaterial for ψ and therefore, from now on, we identify $\overline{\boldsymbol{\mathcal{M}}}$ and $\boldsymbol{\mathcal{M}}$.

Step 3: Let $\Delta = \Delta_1 \cup \Delta_2$, where
- Δ_1 distinguishes the set $T = \{MX^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}$ and $|\Delta_1| \leq |T| - 1$ holds (cf. Step 1 above).
- Δ_2 satisfies $|J \cap \Delta_2| \geq \min(3, |J|)$, for every $J \in \{MX^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}$. Plainly, we can assume that $|\Delta_2| \leq 3 \cdot |\mathcal{V}_1' \cup \mathcal{V}_1^F|$.

We then initialize D^* by putting

$$D^* := \{Mx : x \text{ in } \mathcal{V}_0'\} \cup \Delta\,.$$

Step 4: Let $\psi_1, \ldots, \psi_r$ be the conjuncts of ψ. To each conjunct ψ_i of the form $(\forall Z_{i,h_1}^1) \ldots (\forall Z_{i,h_{m_i}}^1)\varphi_i$ we associate the collection $\varphi_{i,k_1}, \ldots, \varphi_{i,k_{\ell_i}}$ of atomic formulae of the form $(\forall z_1) \ldots (\forall z_n)\varphi_0$ present in the matrix of ψ_i, and call the variables $Z_{i,h_1}^1, \ldots, Z_{i,h_{m_i}}^1$ the *arguments of* $\varphi_{i,k_1}, \ldots, \varphi_{i,k_{\ell_i}}$. Let us put

$$\Phi = \{\varphi_{i,k_j} : 1 \leq j \leq \ell_i \text{ and } 1 \leq i \leq r\}.$$

Then, for each $\varphi \in \Phi$ of the form $(\forall z_1) \ldots (\forall z_n)\varphi_0$ having $Z_1^1, \ldots, Z_m^1$ as arguments, and for each ordered m-tuple $(X_{h_1}^1, \ldots, X_{h_m}^1)$ of variables in $\mathcal{V}_1' \cup \mathcal{V}_1^F$, if $M(\varphi_0)^{Z_1^1, \ldots, Z_m^1}_{X_{h_1}^1, \ldots, X_{h_m}^1} = \mathbf{false}$ we insert in D^* elements $u_1, \ldots, u_n \in D$ such that

$$M[z_1/u_1, \ldots, z_n/u_n](\varphi_0)^{Z_1^1, \ldots, Z_m^1}_{X_{h_1}^1, \ldots, X_{h_m}^1} = \mathbf{false}\,,$$

otherwise we leave D^* unchanged.

Relativized interpretations. We introduce the notion of *relativized interpretation*, to be used together with the domain D^* constructed above, to define, out of a model $\boldsymbol{\mathcal{M}} = (D, M)$ for a $4LQS^R$-formula ψ, a finite interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ of bounded size satisfying ψ as well.

Definition 1. *Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $4LQS^R$-interpretation. Let D^*, $\mathcal{V}'_1, \mathcal{V}^F_1$, and $\mathcal{V}'_2$ be as above, and let $d^* \in D^*$. The* relativized *interpretation* $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1, \mathcal{V}^F_1, \mathcal{V}'_2)$ *of $\boldsymbol{\mathcal{M}}$ with respect to D^*, d^*, $\mathcal{V}'_1$, $\mathcal{V}^F_1$, and $\mathcal{V}'_2$ is the interpretation (D^*, M^*) such that*

$$
\begin{aligned}
M^*x &= \begin{cases} Mx\,, & \text{if } Mx \in D^* \\ d^*\,, & \text{otherwise}\,, \end{cases} \\
M^*X^1 &= MX^1 \cap D^*\,, \\
M^*X^2 &= ((MX^2 \cap \mathrm{pow}(D^*)) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)\}) \\
&\qquad \cup \{M^*X^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1),\ MX^1 \in MX^2\}\,, \\
M^*\langle x, y\rangle &= \{\{M^*x\}, \{M^*x, M^*y\}\}\,, \\
M^*X^3 &= ((MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^*X^2 : X^2 \in \mathcal{V}'_2\})\,, \\
&\qquad \cup \{M^*X^2 : X^2 \in \mathcal{V}'_2,\ MX^2 \in MX^3\}\,.
\end{aligned}
$$

Concerning M^*X^2 and M^*X^3, we observe that they have been defined in such a way that all the membership relations between variables of ψ of sorts 2 and 3 are the same in both the interpretations $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}^*$. This fact will be proved in the next section.

For ease of notation, we will often omit the reference to the element $d^* \in D^*$ and write simply $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{V}'_1, \mathcal{V}^F_1, \mathcal{V}'_2)$ in place of $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1, \mathcal{V}^F_1, \mathcal{V}'_2)$, when d^* is clear from the context.

The following useful properties are immediate consequences of the construction of D^*:

(A) if $MX^1 \neq MY^1$, then $(MX^1 \triangle MY^1) \cap D^* \neq \emptyset$,[2]
(B) if $MX^2 \neq MY^2$, there is a $J \in (MX^2 \triangle MY^2) \cap \{MX^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_F)\}$ such that $J \cap D^* \neq \emptyset$,
(C) if $M\langle x, y\rangle \neq MX^2$, there is a $J \in (MX^2 \triangle M\langle x, y\rangle) \cap \{MX^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_F)\}$ such that $J \cap D^* \neq \emptyset$, and if $J \in MX^2$, $J \cap D^* \neq \{Mx\}$ and $J \cap D^* \neq \{Mx, My\}$,

for any $x, y \in \mathcal{V}'_0$, $X^1, Y^1 \in \mathcal{V}'_1$, and $X^2, Y^2 \in \mathcal{V}'_2$.

3.3 Soundness of the relativization

Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $4LQS^R$-interpretation satisfying a given $4LQS^R$-formula ψ, and let D^*, $\mathcal{V}'_1$, $\mathcal{V}^F_1$, $\mathcal{V}'_2$, and $\boldsymbol{\mathcal{M}}^*$ be defined as above. The main result of this section is Theorem 1 which states that if $\boldsymbol{\mathcal{M}}$ satisfies ψ, then $\boldsymbol{\mathcal{M}}^*$ satisfies ψ as well. The proof of Theorem 1 exploits the technical Lemmas 1, 2, 3, 4, and 5 below. In particular, Lemma 1 states that $\boldsymbol{\mathcal{M}}$ satisfies a quantifier-free atomic formula φ fulfilling conditions (A), (B), and (C), if and only if $\boldsymbol{\mathcal{M}}^*$ satisfies φ too. Lemmas 2, 3, and 4 claim that

[2] We recall that for any sets s and t, $s \triangle t$ denotes the symmetric difference of s and of t, namely the set $(s \setminus t) \cup (t \setminus s)$.

suitably constructed variants of $\boldsymbol{\mathcal{M}}^*$ and the small models resulting by applying the construction of Section 3.2 to the corresponding variants of $\boldsymbol{\mathcal{M}}$ can be considered identical. Finally, Lemma 5, stating that if $\boldsymbol{\mathcal{M}}$ satisfies a quantified conjunction of ψ, then $\boldsymbol{\mathcal{M}}^*$ satisfies it as well, is proved by applying Lemmas 1, 2, 3, and 4.

Proofs of Lemmas 1, 2, 3, and 4 are routine and can be found in Appendices A.1, A.2, A.3, and A.4, respectively.

Lemma 1. *The following statements hold:*

(a) $\boldsymbol{\mathcal{M}}^* \models x = y$ *iff* $\boldsymbol{\mathcal{M}} \models x = y$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$*;*
(b) $\boldsymbol{\mathcal{M}}^* \models x \in X^1$ *iff* $\boldsymbol{\mathcal{M}} \models x \in X^1$*, for all* $X^1 \in \mathcal{V}_1$ *and* $x \in \mathcal{V}_0$ *such that* $Mx \in D^*$*;*
(c) $\boldsymbol{\mathcal{M}}^* \models X^1 = Y^1$ *iff* $\boldsymbol{\mathcal{M}} \models X^1 = Y^1$*, for all* $X^1, Y^1 \in \mathcal{V}_1$ *such that condition (A) holds;*
(d) $\boldsymbol{\mathcal{M}}^* \models X^1 \in X^2$ *iff* $\boldsymbol{\mathcal{M}} \models X^1 \in X^2$*, for all* $X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_F')$*,* $X^2 \in \mathcal{V}_2$*;*
(e) $\boldsymbol{\mathcal{M}}^* \models X^2 = Y^2$ *iff* $\boldsymbol{\mathcal{M}} \models X^2 = Y^2$*, for all* $X^2, Y^2 \in \mathcal{V}_2$ *such that condition (B) holds;*
(f) $\boldsymbol{\mathcal{M}}^* \models \langle x, y\rangle = X^2$ *iff* $\boldsymbol{\mathcal{M}} \models \langle x, y\rangle = X^2$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$ *and* $X^2 \in \mathcal{V}_2$ *such that condition (C) holds;*
(g) $\boldsymbol{\mathcal{M}}^* \models \langle x, y\rangle \in X^3$ *iff* $\boldsymbol{\mathcal{M}} \models \langle x, y\rangle \in X^3$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$ *and* $X^2 \in \mathcal{V}_2$ *such that condition (C) holds;*
(h) $\boldsymbol{\mathcal{M}}^* \models X^2 \in X^3$ *iff* $\boldsymbol{\mathcal{M}} \models X^2 \in X^3$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$ *and* $X^2 \in \mathcal{V}_2$ *such that conditions (B) and (C) hold.* □

In view of the next technical lemmas, we introduce the following notations. Let $u_1, \ldots, u_n \in D^*$, $U_1^1, \ldots, U_m^1 \in \mathrm{pow}(D^*)$, and $U_1^2, \ldots, U_p^2 \in \mathrm{pow}(\mathrm{pow}(D^*))$. Then we put

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{*,z} &= \boldsymbol{\mathcal{M}}^*[z_1/u_1, \ldots, z_n/u_n],\\ \boldsymbol{\mathcal{M}}^{*,Z^1} &= \boldsymbol{\mathcal{M}}^*[Z_1^1/U_1^1, \ldots, Z_m^1/U_m^1],\\ \boldsymbol{\mathcal{M}}^{*,Z^2} &= \boldsymbol{\mathcal{M}}^*[Z_1^2/U_1^2, \ldots, Z_p^2/U_p^2],\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{z,*} &= \mathrm{Rel}(\boldsymbol{\mathcal{M}}^z, D^*, \mathcal{V}_1', \mathcal{V}_1^F, \mathcal{V}_2'),\\ \boldsymbol{\mathcal{M}}^{Z^1,*} &= \mathrm{Rel}(\boldsymbol{\mathcal{M}}^{Z^1}, D^*, \mathcal{V}_1' \cup \{Z_1^1, \ldots, Z_m^1\}, \mathcal{V}_1^F, \mathcal{V}_2'),\\ \boldsymbol{\mathcal{M}}^{Z^2,*} &= \mathrm{Rel}(\boldsymbol{\mathcal{M}}^{Z^2}, D^*, \mathcal{F}^*, \mathcal{V}_1', \mathcal{V}_1^F, \mathcal{V}_2' \cup \{Z_1^2, \ldots, Z_p^2\}).\end{aligned}$$

The next three lemmas claim that, under certain conditions, the following pairs of $4LQS^R$-interpretations $\boldsymbol{\mathcal{M}}^{*,z}$ and $\boldsymbol{\mathcal{M}}^{z,*}$, $\boldsymbol{\mathcal{M}}^{*,Z^1}$ and $\boldsymbol{\mathcal{M}}^{Z^1,*}$, $\boldsymbol{\mathcal{M}}^{*,Z^2}$ and $\boldsymbol{\mathcal{M}}^{Z^2,*}$ can be identified.

Lemma 2. *Let* $u_1, \ldots, u_n \in D^*$*, and let* $z_1, \ldots, z_n \in \mathcal{V}_0$*. Then, for every* $x, y \in \mathcal{V}_0$*,* $X^1 \in \mathcal{V}_1$*,* $X^2 \in \mathcal{V}_2$*,* $X^3 \in \mathcal{V}_3$*, we have:*

(i) $M^{*,z}x = M^{z,*}x$*,*
(ii) $M^{*,z}X^1 = M^{z,*}X^1$*,*

(iii) $M^{*,z}X^2 = M^{z,*}X^2$,
(iv) $M^{*,z}X^3 = M^{z,*}X^3$. □

Lemma 3. *Let* $Z_1^1, \ldots, Z_m^1 \in \mathcal{V}_1 \setminus (\mathcal{V}_1' \cup \mathcal{V}_1^F)$ *and* $U_1^1, \ldots, U_m^1 \in \mathrm{pow}(D^*) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}$. *Then, the* $4LQS^R$*-interpretations* $\boldsymbol{\mathcal{M}}^{*,Z^1}$ *and* $\boldsymbol{\mathcal{M}}^{Z^1,*}$ *coincide.* □

Lemma 4. *Let* $Z_1^2, \ldots, Z_p^2 \in \mathcal{V}_2 \setminus \mathcal{V}_2'$ *and* $U_1^2, \ldots, U_p^2 \in \mathrm{pow}(\mathrm{pow}(D^*)) \setminus \{M^*X^2 : X^2 \in \mathcal{V}_2'\}$. *Then the* $4LQS^R$*-interpretations* $\boldsymbol{\mathcal{M}}^{*,Z^2}$ *and* $\boldsymbol{\mathcal{M}}^{Z^2,*}$ *coincide.* □

The following lemma proves that satisfiability is preserved in the case of quantified atomic formulae.

Lemma 5. *Let* $(\forall z_1) \ldots (\forall z_n)\varphi_0$, $(\forall Z_1^1) \ldots (\forall Z_m^1)\varphi_1$, $(\forall Z_1^2) \ldots (\forall Z_p^2)\varphi_2$, *and* $(\forall Z^2)(Z^2 \in X^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2))$ *be conjuncts of* ψ. *Then*

(i) if $\boldsymbol{\mathcal{M}} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$, *then* $\boldsymbol{\mathcal{M}}^* \models (\forall z_1) \ldots (\forall z_n)\varphi_0$;
(ii) if $\boldsymbol{\mathcal{M}} \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$, *then* $\boldsymbol{\mathcal{M}}^* \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$;
(iii) if $\boldsymbol{\mathcal{M}} \models (\forall Z_1^2) \ldots (\forall Z_p^2)\varphi_2$, *then* $\boldsymbol{\mathcal{M}}^* \models (\forall Z_1^2) \ldots (\forall Z_p^2)\varphi_2$;
(iv) if $\boldsymbol{\mathcal{M}} \models (\forall Z^2)(Z^2 \in X^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2))$, *then* $\boldsymbol{\mathcal{M}}^* \models (\forall Z^2)(Z^2 \in X^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2))$.

Proof. (i) Assume by contradiction that there exist $u_1, \ldots, u_n \in D^*$ such that $\boldsymbol{\mathcal{M}}^{*,z} \not\models \varphi_0$. Then, there must be an atomic formula φ_0' in φ_0 that is interpreted differently in $\boldsymbol{\mathcal{M}}^{*,z}$ and in $\boldsymbol{\mathcal{M}}^z$. Recalling that φ_0 is a propositional combination of quantifier-free atomic formulae of any level, we can suppose that φ_0' is $X^2 = Y^2$ and, without loss of generality, assume that $\boldsymbol{\mathcal{M}}^{*,z} \not\models X^2 = Y^2$. Then $M^{*,z}X^2 \neq M^{*,z}Y^2$, so that, by Lemma 2, $M^{z,*}X^2 \neq M^{z,*}Y^2$. Then, Lemma 1 yields $M^zX^2 \neq M^zY^2$, a contradiction. The other cases are proved in an analogous way.

(ii) This case can proved much along the same lines as the proof of case (ii) of Lemma 4 in [7]. Here, one has only to take care of the fact that the collection of relevant variables of sort 1 for ψ are not just the variables occurring free in ψ, namely the ones in $\mathcal{V}_1'$, but also the variables in $\mathcal{V}_1^F$, introduced to denote the elements distinguishing the sets M^*X^2, for $X^2 \in \mathcal{V}_2'$.

(iii) The proof is carried out as in case (ii).

(iv) Assume by contradiction that there exists a $U \in \mathrm{pow}(\mathrm{pow}(D^*))$ such that $\boldsymbol{\mathcal{M}}^{*,Z^2} \not\models (Z^2 \in X^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2))$. We can distinguish two cases:

1. If there is a $X^2 \in \mathcal{V}_2'$ such that $M^*X^2 = U$, then $\boldsymbol{\mathcal{M}}^* \not\models (X^2 \in X^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2))$ and either $X^2 \in X^3$ or $\neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2)$ must be interpreted differently in $\boldsymbol{\mathcal{M}}^*$ and in $\boldsymbol{\mathcal{M}}$.
   By Lemma 1, $X^2 \in X^3$ is interpreted in the same way in $\boldsymbol{\mathcal{M}}^*$ and in $\boldsymbol{\mathcal{M}}$. By case (i) of this lemma, if $\boldsymbol{\mathcal{M}}^* \models \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2)$ then $\boldsymbol{\mathcal{M}} \models \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2)$. Thus, the only case to be considered is when $\boldsymbol{\mathcal{M}}^* \models (\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2)$. Assume that $\boldsymbol{\mathcal{M}} \models \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2)$. Then MX^2 must be a pair $\{\{u\}, \{u, v\}\}$, for some $u, v \in D$. But then by the construction of the universe D^*, we have $u, v \in D^*$, contradicting the hypothesis that $\boldsymbol{\mathcal{M}}^* \models (\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = X^2)$.

2. If $U \neq M^* X^2$, for every $X^2 \in \mathcal{V}'_2$, either $Z^2 \in X^3$ or $\neg(\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$ has to be interpreted in a different way in $\boldsymbol{\mathcal{M}}^{*,Z^2}$ and in $\boldsymbol{\mathcal{M}}^{Z^2}$.
   By Lemmas 4 and 1, and by case (i) of this lemma, $Z^2 \in X^3$ is evaluated in the same way in $\boldsymbol{\mathcal{M}}^{*,Z^2}$ and in $\boldsymbol{\mathcal{M}}^{Z^2}$, and if $\boldsymbol{\mathcal{M}}^{*,Z^2} \models \neg(\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$ then $\boldsymbol{\mathcal{M}}^{Z^2} \models \neg(\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$. The only case that still has to be analyzed is when $\boldsymbol{\mathcal{M}}^{*,Z^2} \models (\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$. By Lemma 4, $\boldsymbol{\mathcal{M}}^{Z^2,*} \models (\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$. Let us assume that $\boldsymbol{\mathcal{M}}^{Z^2} \not\models (\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$. Then U must be a pair $\{\{u\}, \{u, v\}\}$, $u, v \in D$. Since $U \in \mathrm{pow}(\mathrm{pow}(D^*))$, then $u, v \in D^*$, contradicting that $\boldsymbol{\mathcal{M}}^{Z^2,*} \models (\forall z_1, z_2)\neg(\langle z_1, z_2 \rangle = Z^2)$. ■

Next, we can state our main result.

Theorem 1. *Let $\boldsymbol{\mathcal{M}}$ be a $4LQS^R$-interpretation satisfying ψ. Then $\boldsymbol{\mathcal{M}}^* \models \psi$.*

Proof. We have to prove that $\boldsymbol{\mathcal{M}}^* \models \psi'$ for each literal ψ' occurring in ψ. Each ψ' must be of one of the types introduced in Section 3.1. By applying Lemmas 1 or 5 to every ψ' (according to its type) we obtain the thesis. ■

From the above reduction and relativization steps, it is not hard to derive the following result:

Corollary 1. *The fragment $4LQS^R$ enjoys a small model property (and therefore its satisfiability problem is solvable).* □

3.4 Complexity issues

Let $(4LQS^R)^k$ be the sublanguage of $4LQS^R$ in which the quantifier prefixes of quantified atoms of level 2 have length not exceeding k. Then the following result holds.

Lemma 6. *The satisfiability problem for $(4LQS^R)^k$ is* NP*-complete, for any $k \in \mathbb{N}$.*

Proof. NP-hardness is trivially proved by reducing an instance of the satisfiability problem of propositional logic to our problem.

To prove that our problem is in NP, we reason as follows. Let φ be a satisfiable $(4LQS^R)^k$-formula. Let φ_{DNF} be a disjunctive normal form of φ. Then there is a disjunct ψ of φ_{DNF} that is satisfied by a $(4LQS^R)^k$-interpretation $\boldsymbol{\mathcal{M}} = (D, M)$. After the normalization step, ψ is a normalized $(4LQS^R)^k$-conjunction satisfied by $\boldsymbol{\mathcal{M}}$ and, according to the procedure of Section 3.2, we can construct a small interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ satisfying ψ and such that $|D^*|$ is polynomial in the size of ψ. This can be shown by recalling that $|\mathcal{F}_1| \leq |S| - 1 \leq |\mathcal{V}'_2| - 1$ and that $|\mathcal{F}_2| \leq 3|\mathcal{V}'_2|$ (cf. Step 1 of the procedure in Section 3.2). Thus, clearly, $|\mathcal{F}| \leq 4|\mathcal{V}'_2| - 1$. Analogously, from Step 3, $|\Delta| \leq 4(|\mathcal{V}'_1| + (4|\mathcal{V}'_2| - 1)) - 1$, and $|D^*|$ (in the initialization phase) is bounded by $|\mathcal{V}'_0| + 4|\mathcal{V}'_1| + 16|\mathcal{V}'_2| - 5$. Finally, after Step 4, if we let L_n denote the maximal length of the quantifier prefix of $\varphi = (\forall z_1) \ldots (\forall z_n)\varphi_0$, with φ varying in Φ, then $|D^*| \leq |\mathcal{V}'_0| + 4|\mathcal{V}'_1| + 16|\mathcal{V}'_2| - 5 + ((|\mathcal{V}'_1| + 4|\mathcal{V}'_2| - 1)^k L_n)|\Phi|$. Thus the size of D^* is polynomial in the size of ψ. Since $\boldsymbol{\mathcal{M}}^* \models \psi$ can be verified in polynomial time and the size of ψ is polynomial w.r.t. the size of φ, it results that the satisfiability problem for $(4LQS^R)^k$ is in NP, and therefore it is NP-complete. ■

4 Expressiveness of the language $4LQS^R$

As discussed in [7], $4LQS^R$ can express a restricted variant of the set former, which in turn allows to express other significant set operators such as binary union, intersection, set difference, the singleton operator, the powerset operator (over subsets of the universe only), etc. More specifically, atomic formulae of type $X^1 = \{z : \varphi(z)\}$ or $X^i = \{X^{i-1} : \varphi(X^{i-1})\}$, for $i \in \{2, 3\}$, can be expressed in $4LQS^R$ by the formulae

$$(\forall z)(z \in X^1 \leftrightarrow \varphi(z)) \tag{2}$$

$$(\forall X^{i-1})(X^{i-1} \in X^i \leftrightarrow \varphi(X^{i-1})) \tag{3}$$

provided that they satisfy the syntactic constraints of $4LQS^R$.

Since $4LQS^R$ is a superlanguage of *3LQS*^{R}, as shown in [7] $4LQS^R$ can express the stratified syllogistic $2LS$ and the sublanguage $3LSSP$ of $3LSSPU$ not involving the set-theoretic construct of general union. We recall that $3LSSPU$ admits variables of three sorts and, besides the usual set-theoretical constructs, it involves the 'singleton set' operator $\{\cdot\}$, the powerset operator pow, and the general union operator Un.

$3LSSP$ can plainly be decided by the decision procedure presented in [3] for the whole $3LSSPU$.

Other constructs of set theory which are expressible in the $4LQS^R$ formalism, as shown in [7], are:

- the literal $X^2 = \mathrm{pow}_{\leq h}(X^1)$, where $\mathrm{pow}_{\leq h}(X^1)$ denotes the collection of all the subsets of X^1 having at most h elements;
- the literal $X^2 = \mathrm{pow}_{=h}(X^1)$, where $\mathrm{pow}_{=h}(X^1)$ denotes the collection of subsets of X^1 with exactly h elements;
- the unordered Cartesian product $X^2 = X^1_1 \otimes \ldots \otimes X^1_n$;
- the literal $A = \mathrm{pow}^*(X^1_1, \ldots, X^1_n)$, where $\mathrm{pow}^*(X^1_1, \ldots, X^1_n)$ is a variant of the powerset which denotes the collection

$$\{Z : Z \subseteq \bigcup_{i=1}^{n} X^1_i \text{ and } Z \cap X^1_i \neq \emptyset, \text{ for all } 1 \leq i \leq n\}$$

  introduced in [1].

4.1 Other applications of $4LQS^R$

Within the $4LQS^R$ language it is also possible to define binary relations over elements of a domain together with several conditions on them which characterize accessibility relations of well-known modal logics. These formalizations are illustrated in Table 1.

Usual Boolean operations over relations can be defined as shown in Table 2. Within the $4LQS^R$ fragment it is also possible to define the inverse of a given binary relation R^3_1, namely $R^3_2 = (R^3_1)^{-1}$, by means of the $4LQS^R$-formula $(\forall z_1, z_2)(\langle z_1, z_2\rangle \in R^3_1 \leftrightarrow \langle z_2, z_1\rangle \in R^3_2)$.

In the next section we will show how the $4LQS^R$ fragment can be used to formalize some normal modal logics.

Binary relation	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2))$
Reflexive	$(\forall z_1)(\langle z_1, z_1\rangle \in R^3)$
Symmetric	$(\forall z_1, z_2)(\langle z_1, z_2\rangle \in R^3 \rightarrow \langle z_2, z_1\rangle \in R^3)$
Transitive	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_3\rangle \in R^3) \rightarrow \langle z_1, z_3\rangle \in R^3)$
Euclidean	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_1, z_3\rangle \in R^3) \rightarrow \langle z_2, z_3\rangle \in R^3)$
Weakly-connected	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_1, z_3\rangle \in R^3)$ $\rightarrow (\langle z_2, z_3\rangle \in R^3 \vee z_2 = z_3 \vee \langle z_3, z_2\rangle \in R^3))$
Irreflexive	$(\forall z_1)\neg(\langle z_1, z_1\rangle \in R^3)$
Intransitive	$(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_3\rangle \in R^3) \rightarrow \neg\langle z_1, z_3\rangle \in R^3)$
Antisymmetric	$(\forall z_1, z_2)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_1\rangle \in R^3) \rightarrow (z_1 = z_2))$
Asymmetric	$(\forall z_1, z_2)(\langle z_1, z_2\rangle \in R^3 \rightarrow \neg(\langle z_2, z_1\rangle \in R^3))$

Table 1. $4LQS^R$ formalization of conditions of accessibility relations

Intersection	$R^3 = R_1^3 \cap R_2^3$	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow (Z^2 \in R_1^3 \wedge Z^2 \in R_2^3))$
Union	$R^3 = R_1^3 \cup R_2^3$	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow (Z^2 \in R_1^3 \vee Z^2 \in R_2^3))$
Complement	$R_1^3 = \overline{R_2^3}$	$(\forall Z^2)(Z^2 \in R_1^3 \leftrightarrow \neg(Z^2 \in R_2^3))$
Set difference	$R^3 = R_1^3 \setminus R_2^3$	$(\forall Z^2)(Z^2 \in R^3 \leftrightarrow (Z^2 \in R_1^3 \wedge \neg(Z^2 \in R_2^3)))$
Set inclusion	$R_1^3 \subseteq R_2^3$	$(\forall Z^2)(Z^2 \in R_1^3 \rightarrow Z^2 \in R_2^3)$

Table 2. $4LQS^R$ formalization of Boolean operations over relations

4.2 Some normal modal logics expressible in $4LQS^R$

The *modal language* L_M is based on a countably infinite set of propositional letters $\mathcal{P} = \{p_1, p_2, \ldots\}$, the classical propositional connectives '$\neg$', '$\wedge$' , and '$\vee$', the modal operators '$\Box$', '$\Diamond$' (and the parentheses). L_M is the smallest set such that $\mathcal{P} \subseteq \mathsf{L}_M$, and such that if $\varphi, \psi \in \mathsf{L}_M$, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\Box\varphi$, $\Diamond\varphi \in \mathsf{L}_M$. Lower case letters like p denote elements of $\mathcal{P}$ and Greek letters like φ and ψ represent formulae of L_M. Given a formula φ of L_M, we indicate with $SubF(\varphi)$ the set of the subformulae of φ. The *modal depth* of a formula φ is the maximum nesting depth of modalities occurring in φ.

A *normal modal logic* is any subset of L_M which contains all the tautologies and the axiom

$$\mathsf{K}: \ \Box(p_1 \rightarrow p_2) \rightarrow (\Box p_1 \rightarrow \Box p_2)\,,$$

and which is closed with respect to modus ponens, substitution, and necessitation (the reader may consult a text on modal logic like [9] for more details).

A *Kripke frame* is a pair $\langle W, R\rangle$ such that W is a nonempty set of possible worlds and R is a binary relation on W called *accessibility relation*. If $R(w, u)$ holds, we say that the world u is accessible from the world w. A *Kripke model* is a triple $\langle W, R, h\rangle$, where $\langle W, R\rangle$ is a Kripke frame and h is a function mapping propositional letters into subsets of W. Thus, $h(p)$ is the set of all the worlds where p is true.

Let $K = \langle W, R, h\rangle$ be a Kripke model and let w be a world in K. Then, for every $p \in \mathcal{P}$ and for every $\varphi, \psi \in \mathsf{L}_M$, the relation of satisfaction $\models$ is defined as follows:

Axiom	Schema	Condition on R (see Table 1)
T	$\Box p \to p$	Reflexive
5	$\Diamond p \to \Box\Diamond p$	Euclidean
B	$p \to \Box\Diamond p$	Symmetric
4	$\Box p \to \Box\Box p$	Transitive
D	$\Box p \to \Diamond p$	Serial: $(\forall w)(\exists u)R(w,u)$

Table 3. Axioms of normal modal logics

- $K, w \models p$ iff $w \in h(p)$;
- $K, w \models \varphi \vee \psi$ iff $K, w \models \varphi$ or $K, w \models \psi$;
- $K, w \models \varphi \wedge \psi$ iff $K, w \models \varphi$ and $K, w \models \psi$;
- $K, w \models \neg\varphi$ iff $K, w \not\models \varphi$;
- $K, w \models \Box\varphi$ iff $K, w' \models \varphi$, for every $w' \in W$ such that $(w, w') \in R$;
- $K, w \models \Diamond\varphi$ iff there is a $w' \in W$ such that $(w, w') \in R$ and $K, w' \models \varphi$.

A formula φ is said to be *satisfied* at w in K if $K, w \models \varphi$; φ is said to be *valid* in K (and we write $K \models \varphi$), if $K, w \models \varphi$, for every $w \in W$.

The smallest normal modal logic is K, which contains only the modal axiom K and whose accessibility relation R can be any binary relation. The other normal modal logics admit together with K other modal axioms drawn from the ones in Table 3.

Translation of a normal modal logic into the $4LQS^R$ language is based on the semantics of propositional and modal operators. For any normal modal logic, the formalization of the semantics of modal operators depends on the axioms that characterize the logic. In the case of the logics S5 and K45, proved to be NP-complete in [11], and introduced next, the $4LQS^R$ formalization of the modal formulae $\Box\varphi$ and $\Diamond\varphi$ turns out to be straightforward and thus these logics can be entirely translated into the $4LQS^R$ language. This is illustrated in what follows.

The logic S5. Modal logic S5 is the strongest normal modal system. It can be obtained from the logic K in several ways. One of them consists in adding axioms T and **5** from Table 3 to the logic K. Given a formula φ, a Kripke model $K = \langle W, R, h \rangle$, and a world $w \in W$, the semantics of the modal operators can be defined as follows:

- $K, w \models \Box\varphi$ iff $K, v \models \varphi$, for every $v \in W$,
- $K, w \models \Diamond\varphi$ iff $K, v \models \varphi$, for some $v \in W$.

This makes it possible to translate a formula φ of S5 into the $4LQS^R$ language.

For the purpose of simplifying the definition of the translation function τ_{S5} given below, the concept of "empty formula" is introduced, to be denoted by Λ, and not interpreted in any particular way. The only requirement on Λ needed for the definition given next is that $\Lambda \wedge \psi$ and $\psi \wedge \Lambda$ are to be considered as syntactic variations of ψ, for any $4LQS^R$-formula ψ.

For every propositional letter p, let $\tau^1_{\mathsf{S5}}(p) = X^1_p$, where $X^1_p \in \mathcal{V}_1$, and let τ^2_{S5} : $\mathsf{S5} \to 4LQS^R$ be the function defined recursively as follows:

- $\tau^2_{\mathsf{S5}}(p) = \Lambda$,
- $\tau^2_{\mathsf{S5}}(\neg\varphi) = (\forall z)(z \in X^1_{\neg\varphi} \leftrightarrow \neg(z \in X^1_{\varphi})) \wedge \tau^2_{\mathsf{S5}}(\varphi)$,
- $\tau^2_{\mathsf{S5}}(\varphi_1 \wedge \varphi_2) = (\forall z)(z \in X^1_{\varphi_1\wedge\varphi_2} \leftrightarrow (z \in X^1_{\varphi_1} \wedge z \in X^1_{\varphi_2})) \wedge \tau^2_{\mathsf{S5}}(\varphi_1) \wedge \tau^2_{\mathsf{S5}}(\varphi_2)$,
- $\tau^2_{\mathsf{S5}}(\varphi_1 \vee \varphi_2) = (\forall z)(z \in X^1_{\varphi_1\vee\varphi_2} \leftrightarrow (z \in X^1_{\varphi_1} \vee z \in X^1_{\varphi_2})) \wedge \tau^2_{\mathsf{S5}}(\varphi_1) \wedge \tau^2_{\mathsf{S5}}(\varphi_2)$,
- $\tau^2_{\mathsf{S5}}(\Box\varphi) =$
  $(\forall z)(z \in X^1_{\varphi}) \rightarrow (\forall z)(z \in X^1_{\Box\varphi}) \wedge \neg(\forall z)(z \in X^1_{\varphi}) \rightarrow (\forall z)\neg(z \in X^1_{\Box\varphi}) \wedge \tau^2_{\mathsf{S5}}(\varphi)$,
- $\tau^2_{\mathsf{S5}}(\Diamond\varphi) =$
  $\neg(\forall z)\neg(z \in X^1_{\varphi}) \rightarrow (\forall z)(z \in X^1_{\Diamond\varphi}) \wedge (\forall z)\neg(z \in X^1_{\varphi}) \rightarrow (\forall z)\neg(z \in X^1_{\Diamond\varphi}) \wedge \tau^2_{\mathsf{S5}}(\varphi)$,

where Λ is the empty formula and $X^1_{\neg\varphi}, X^1_{\varphi}, X^1_{\varphi_1\wedge\varphi_2}, X^1_{\varphi_1\vee\varphi_2}, X^1_{\varphi_1}, X^1_{\varphi_2} \in \mathcal{V}_1$.

Finally, for every φ in S5, if φ is a propositional letter in $\mathcal{P}$ we put $\tau_{\mathsf{S5}}(\varphi) = \tau^1_{\mathsf{S5}}(\varphi)$, otherwise $\tau_{\mathsf{S5}}(\varphi) = \tau^2_{\mathsf{S5}}(\varphi)$.

Even though the accessibility relation R is not used in the translation, we can give its formalization in the $4LQS^R$ fragment. Let U be defined so that $(\forall z)(z \in U)$, then R can be defined in the following two ways:

1. as a variable of sort 2, R^2, such that
   $(\forall Z^1)(Z^1 \in R^2 \leftrightarrow (Z^1 \in \mathrm{pow}_{=1}(U) \vee Z^1 \in \mathrm{pow}_{=2}(U)))$,
2. as a variable of sort 3, R^3, such that
   $(\forall Z^2)(Z^2 \in R^3 \leftrightarrow \neg(\forall z_1, z_2)\neg(\langle z_1, z_2\rangle = Z^2)) \wedge (\forall z_1)(\langle z_1, z_1\rangle \in R^3)$
   $\wedge(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_1, z_3\rangle \in R^3) \rightarrow \langle z_2, z_3\rangle \in R^3)$.

Correctness of the above translation is guaranteed by the following lemma, whose proof can be found in Appendix A.5.

Lemma 7. *For every formula φ of the logic* S5*, φ is satisfiable in a model $K = \langle W, R, h\rangle$ iff there is a $4LQS^R$-interpretation satisfying $x \in X_{\varphi}$.* □

It can be checked that $\tau_{\mathsf{S5}}(\varphi)$ is polynomial in the size of φ and that its satisfiability can be verified in nondeterministic polynomial time since it belongs to $(4LQS^R)^1$. Consequently, the decision algorithm presented in this paper together with the translation function introduced above can be considered an optimal procedure (in terms of its computational complexity class) to decide the satisfiability of any formula φ of S5. Moreover, it can be noticed that if we apply the first definition of R, S5 can be expressed by the language $3LQS^R$ presented in [7].

The logic K45. The normal modal logic K45 is obtained from the logic K by adding axioms **4** and **5** described in Table 3 to K. Semantics of the modal operators □ and ◊ for the logic K45 can be described as follows. Given a formula φ of K45 and a Kripke model $K = \langle W, R, h\rangle$,

- $K \models \Box\varphi$ iff $K, v \models \varphi$, for every $v \in W$ s.t. there is a $w' \in W$ with $(w', v) \in R$,

– $K \models \Diamond\varphi$ iff $K, v \models \varphi$, for some $v \in W$ s.t. there is a $w' \in W$ with $(w', v) \in R$.

It is convenient, before translating K45 into the $4LQS^R$ fragment, to introduce the $4LQS^R$-formula which describes the semantics of the accessibility relation R of the logic K45:

$$
\begin{aligned}
&(\forall Z^2)(Z^2 \in R^3 \leftrightarrow \neg(\forall z_1)(\forall z_2)\neg(\langle z_1, z_2\rangle = Z^2))\\
&\quad \wedge(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_2, z_3\rangle \in R^3) \rightarrow \langle z_1, z_3\rangle \in R^3)\\
&\quad\quad \wedge(\forall z_1, z_2, z_3)((\langle z_1, z_2\rangle \in R^3 \wedge \langle z_1, z_3\rangle \in R^3) \rightarrow \langle z_2, z_3\rangle \in R^3).
\end{aligned}
$$

The transformation function $\tau_{\mathsf{K45}} : \mathsf{K45} \rightarrow 4LQS^R$ is constructed as for S5. For every $\varphi \in \mathsf{K45}$ we put $\tau_{\mathsf{K45}}(\varphi) = \tau^1_{\mathsf{K45}}(\varphi)$, if φ is a propositional letter and $\tau_{\mathsf{K45}}(\varphi) = \tau^2_{\mathsf{K45}}(\varphi)$ otherwise. $\tau^1_{\mathsf{K45}}(p) = X^1_p$, with $X^1_p \in \mathcal{V}_1$, for every propositional letter p, and $\tau^2_{\mathsf{K45}}(\varphi)$ is defined inductively over the structure of φ. We report the definition of $\tau^2_{\mathsf{K45}}(\varphi)$ only when $\varphi = \Box\psi$ and $\varphi = \Diamond\psi$, as the other cases are identical to $\tau^2_{\mathsf{S5}}(\varphi)$, defined in the previous section:

– $\tau^2_{\mathsf{K45}}(\Box\psi) = (\forall z_1)((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \rightarrow z_1 \in X^1_\psi) \rightarrow (\forall z)(z \in X^1_{\Box\psi})$
$\wedge\neg(\forall z_1)\neg((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \wedge \neg(z_1 \in X^1_\psi)) \rightarrow (\forall z)\neg(z \in X^1_{\Box\psi}) \wedge \tau^2_{\mathsf{K45}}(\psi)$;

– $\tau^2_{\mathsf{K45}}(\Diamond\psi) = \neg(\forall z_1)\neg((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \wedge z_1 \in X^1_\psi) \rightarrow (\forall z)(z \in X^1_{\Diamond\psi})$
$\wedge(\forall z_1)((((\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \vee \neg(z_1 \in X^1_\psi))) \rightarrow (\forall z)\neg(z \in X^1_{\Diamond\psi}) \wedge \tau^2_{\mathsf{K45}}(\psi)$.

The following lemma, proved in Appendix A.6, shows the correctness of the translation.

Lemma 8. *For every formula φ of the logic τ_{K45}, φ is satisfiable in a model $K = \langle W, R, h\rangle$ iff there is a $4LQS^R$-interpretation satisfying $x \in X_\varphi$.* □

As for S5, it can be checked that $\tau_{\mathsf{K45}}(\varphi)$ is polynomial in the size of φ and that its satisfiability can be verified in nondeterministic polynomial time since it belongs to the sublanguage $(4LQS^R)^1$ of $4LQS^R$. Thus, the decision algorithm we have presented and the translation function introduced above represent an optimal procedure (in terms of its computational complexity class) to decide satisfiability of any formula φ of K45.

5 Conclusions

We have presented a decidability result for the satisfiability problem for the fragment $4LQS^R$ of multi-sorted stratified syllogistic embodying variables of four sorts and a restricted form of quantification. As the semantics of the modal formulae $\Box\varphi$ and $\Diamond\varphi$ in the modal logics S5 and K45 can be easily formalized in $4LQS^R$, it follows that $4LQS^R$ can express both logics S5 and K45.

Currently, in the case of modal logics characterized by having a liberal accessibility relation like K, we are not able to translate the modal formulae $\Box\varphi$ and $\Diamond\varphi$ in $4LQS^R$. The same problem concerns also the composition operation on binary relations and the set-theoretical operation of general union. We intend to investigate such a question more in depth and verify whether a formalization of these constructs is still possible in $4LQS^R$ or if an extension of the language $4LQS^R$ is required. In the same direction, we aim at finding a characterization of the conditions that an accessibility relation has to fulfil in order for a modal logic to be formalized in $4LQS^R$. We also intend to find classes of modal formulae with bounded modal nesting and multi-modal logics that can be embedded in the $4LQS^R$ framework. Finally, since $4LQS^R$ is able to express Boolean operations on relations, we plan to investigate the possibility of translating fragments of Boolean modal logics into $4LQS^R$.

References

1. D. Cantone. Decision procedures for elementary sublanguages of set theory: X. Multilevel syllogistic extended by the singleton and powerset operators. *Journal of Automated Reasoning*, volume 7, number 2, pages 193–230, Kluwer Academic Publishers, Hingham, MA, USA, 1991.
2. D. Cantone and V. Cutello. A decidable fragment of the elementary theory of relations and some applications. In *ISSAC '90: Proceedings of the international symposium on Symbolic and algebraic computation*, pages 24–29, New York, NY, USA, 1990. ACM Press.
3. D. Cantone and V. Cutello. Decision procedures for stratified set-theoretic syllogistics. In Manuel Bronstein, editor, *Proceedings of the 1993 International Symposium on Symbolic and Algebraic Computation, ISSAC'93 (Kiev, Ukraine, July 6-8, 1993)*, pages 105–110, New York, 1993. ACM Press.
4. D. Cantone, V. Cutello, and J. T. Schwartz. Decision problems for Tarski and Presburger arithmetics extended with sets. In *CSL '90: Proceedings of the 4th Workshop on Computer Science Logic*, pages 95–109, London, UK, 1991. Springer-Verlag.
5. D. Cantone and A. Ferro Techniques of computable set theory with applications to proof verification. *Comm. Pure Appl. Math.*, pages 901–945, vol. XLVIII, 1995. Wiley.
6. D. Cantone, A. Ferro, and E. Omodeo. *Computable set theory.* Clarendon Press, New York, NY, USA, 1989.
7. D. Cantone and M. Nicolosi Asmundo. On the satisfiability problem for a 3-level quantified syllogistic. In *Proceedings of CEDAR'08".* Sydney, Australia, 11 August, 2008.
8. D. Cantone, E. Omodeo, and A.Policriti. *Set Theory for Computing - From decision procedures to declarative programming with sets.* Springer-Verlag, Texts and Monographs in Computer Science, 2001.
9. P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, Cambridge Tracts in Theoretical Computer Science, 2001.
10. A. Ferro and E.G. Omodeo. *An efficient validity test for formulae in extensional two-level syllogistic.* Le Matematiche, 33:130–137, 1978.
11. R. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal of Computing,* 6: 467-480, 1977.

A Proofs of some lemmas

A.1 Proof of Lemma 1

Lemma 1. *The following statements hold:*

(a) $\mathcal{M}^* \models x = y$ *iff* $\mathcal{M} \models x = y$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$*;*
(b) $\mathcal{M}^* \models x \in X^1$ *iff* $\mathcal{M} \models x \in X^1$*, for all* $X^1 \in \mathcal{V}_1$ *and* $x \in \mathcal{V}_0$ *such that* $Mx \in D^*$*;*
(c) $\mathcal{M}^* \models X^1 = Y^1$ *iff* $\mathcal{M} \models X^1 = Y^1$*, for all* $X^1, Y^1 \in \mathcal{V}_1$ *such that condition (A) holds;*
(d) $\mathcal{M}^* \models X^1 \in X^2$ *iff* $\mathcal{M} \models X^1 \in X^2$*, for all* $X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}'_F)$*,* $X^2 \in \mathcal{V}_2$*;*
(e) $\mathcal{M}^* \models X^2 = Y^2$ *iff* $\mathcal{M} \models X^2 = Y^2$*, for all* $X^2, Y^2 \in \mathcal{V}_2$ *such that condition (B) holds;*
(f) $\mathcal{M}^* \models \langle x, y\rangle = X^2$ *iff* $\mathcal{M} \models \langle x, y\rangle = X^2$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$ *and* $X^2 \in \mathcal{V}_2$ *such that condition (C) holds;*
(g) $\mathcal{M}^* \models \langle x, y\rangle \in X^3$ *iff* $\mathcal{M} \models \langle x, y\rangle \in X^3$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$ *and* $X^2 \in \mathcal{V}_2$ *such that condition (C) holds;*
(h) $\mathcal{M}^* \models X^2 \in X^3$ *iff* $\mathcal{M} \models X^2 \in X^3$*, for all* $x, y \in \mathcal{V}_0$ *such that* $Mx, My \in D^*$ *and* $X^2 \in \mathcal{V}_2$ *such that conditions (B) and (C) hold.*

Proof. (a) Let $x, y \in \mathcal{V}_0$ be such that $Mx, My \in D^*$. Then $M^*x = Mx$ and $M^*y = My$, so we have immediately that $\mathcal{M}^* \models x = y$ iff $\mathcal{M} \models x = y$.

(b) Let $X^1 \in \mathcal{V}_1$ and let $x \in \mathcal{V}_0$ be such that $Mx \in D^*$. Then $M^*x = Mx$, so that $M^*x \in M^*X^1$ iff $Mx \in MX^1 \cap D^*$ iff $Mx \in MX^1$.

(c) If $MX^1 = MY^1$, then plainly $M^*X^1 = M^*Y^1$. On the other hand, if $MX^1 \neq MY^1$, then, by condition (A), $(MX^1 \triangle MY^1) \cap D^* \neq \emptyset$ and thus $M^*X^1 \neq M^*Y^1$.

(d) If $MX^1 \in MX^2$, then $M^*X^1 \in M^*X^2$. On the other hand, suppose by contradiction that $MX^1 \notin MX^2$ and $M^*X^1 \in M^*X^2$. Then, there must necessarily be a $Z^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)$ with $MZ^1 \in MX^2$, $MZ^1 \neq MX^1$, and $M^*X^1 = M^*Z^1$. Since $MZ^1 \neq MX^1$ and $(MZ^1 \triangle MX^1) \cap D^* \neq \emptyset$, by condition (A), we have $M^*X^1 \neq M^*Z^1$, which is a contradiction.

(e) If $MX^2 = MY^2$, then $M^*X^2 = M^*Y^2$. On the other hand, if $MX^2 \neq MY^2$, by condition (B), there is a $J \in (MX^2 \triangle MY^2) \cap \{MX^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)\}$ such that $J \cap D^* \neq \emptyset$. Let $J = MX^1$, for some $X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)$, and suppose without loss of generality that $MX^1 \in MX^2$ and $MX^1 \notin MY^2$. Then, by (d), $M^*X^1 \in M^*X^2$ and $M^*X^1 \notin M^*Y^2$ and hence $M^*X^2 \neq M^*Y^2$.

(f) If $M\langle x, y\rangle = MX^2$, then $M^*\langle x, y\rangle = M^*X^2$. If $M\langle x, y\rangle \neq MX^2$, then there is a $J \in (MX^2 \triangle M\langle x, y\rangle) \cap \{MX^1 : X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)\}$ satisfying the constraints of condition (C). Let $J = MX^1$, for some $X^1 \in (\mathcal{V}'_1 \cup \mathcal{V}^F_1)$, and suppose that $MX^1 \in MX^2$ and $MX^1 \notin M\langle x, y\rangle$. Then $M^*X^1 \in M^*X^2$ and since $M^*X^1 \neq \{Mx\}$ and $M^*X^1 \neq \{Mx, My\}$, it follows that $M^*X^1 \notin M^*\langle x, y\rangle$. On the other hand, if $MX^1 \in M\langle x, y\rangle$ and $MX^1 \notin MX^2$, then either $MX^1 = \{Mx\}$ or $MX^1 = \{Mx, My\}$. In both cases $MX^1 = M^*X^1$ and thus if $MX^1 \notin MX^2$, it plainly follows that $M^*X^1 \notin M^*X^2$.

(g) Let $x, y \in \mathcal{V}_0$ and $X^3 \in \mathcal{V}_3$ be such that $M\langle x, y\rangle \in MX^3$. Then $M^*\langle x, y\rangle \in M^*X^3$. On the other hand, suppose by contradiction that $M\langle x, y\rangle \notin MX^3$ and $M^*\langle x, y\rangle \in M^*X^3$. Then, there must be an $X^2 \in \mathcal{V}'_2$ such that $M^*X^2 \in M^*X^3$, $M^*X^2 = M^*\langle x, y\rangle$, and $MX^2 \neq M\langle x, y\rangle$. But this is impossible by (f).

(h) If $MX^2 \in MX^3$ then $M^*X^2 \in M^*X^3$. Now suppose by contradiction that $MX^2 \notin MX^3$ and that $M^*X^2 \in M^*X^3$. Then, either there is a $Y^2 \in \mathcal{V}_2'$ such that $MX^2 \neq MY^2$ and $M^*X^2 = M^*Y^2$, which is not possible by (e), or there is a $\langle x, y\rangle$, with $x, y \in \mathcal{V}_0$, $Mx, My \in D^*$, such that $MX^2 \neq M\langle x, y\rangle$ and $M^*X^2 = M^*\langle x, y\rangle$, but this is absurd by (f). ■

A.2 Proof of Lemma 2

Lemma 2. *Let $u_1, \ldots, u_n \in D^*$, and let $z_1, \ldots, z_n \in \mathcal{V}_0$. Then, for every $x, y \in \mathcal{V}_0$, $X^1 \in \mathcal{V}_1$, $X^2 \in \mathcal{V}_2$, $X^3 \in \mathcal{V}_3$, we have:*

(i) $M^{*,z}x = M^{z,*}x$,
(ii) $M^{*,z}X^1 = M^{z,*}X^1$,
(iii) $M^{*,z}X^2 = M^{z,*}X^2$,
(iv) $M^{*,z}X^3 = M^{z,*}X^3$.

Proof. (i) Since $u_1, \ldots, u_n \in D^*$, the thesis follows immediately.
(ii) Let $X^1 \in \mathcal{V}_1$, then $M^{*,z}X^1 = M^*X^1 = MX^1 \cap D^* = M^zX^1 \cap D^* = M^{z,*}X^1$.
(iii) Let $X^2 \in \mathcal{V}_2$, then we have the following equalities:

$$\begin{aligned} M^{*,z}X^2 = M^*X^2 &= ((MX^2 \cap \mathrm{pow}(D^*)) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\qquad \cup \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F),\ MX^1 \in MX^2\}, \\ &= ((M^zX^2 \cap \mathrm{pow}(D^*)) \setminus \{M^{z,*}X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\qquad \cup \{M^{z,*}X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F), M^zX^1 \in M^zX^2\} \\ &= M^{z,*}X^2 . \end{aligned}$$

(iv) Let $X^3 \in \mathcal{V}_3$, then the following holds:

$$\begin{aligned} M^{*,z}X^3 = M^*X^3 &= ((MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^*X^2 : X^2 \in \mathcal{V}_2'\}) \\ &\qquad \cup \{M^*X^2 : X^2 \in \mathcal{V}_2', MX^2 \in MX^3\}, \\ &= ((M^zX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^{z,*}X^2 : X^2 \in \mathcal{V}_2'\}) \\ &\qquad \cup \{M^{z,*}X^2 : X^2 \in \mathcal{V}_2', M^zX^2 \in M^zX^3\} \\ &= M^{z,*}X^3 . \end{aligned}$$

■

A.3 Proof of Lemma 3

Lemma 3. *Let $Z^1_1, \ldots, Z^1_m \in \mathcal{V}_1 \setminus (\mathcal{V}_1' \cup \mathcal{V}_1^F)$ and $U^1_1, \ldots, U^1_m \in \mathrm{pow}(D^*) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}$. Then, the $4LQS^R$-interpretations $\boldsymbol{\mathcal{M}}^{*,Z^1}$ and $\boldsymbol{\mathcal{M}}^{Z^1,*}$ coincide.*

Proof. We prove the lemma by showing that $\mathcal{M}^{*,Z^1}$ and $\mathcal{M}^{Z^1,*}$ agree over variables of all sorts.

1. Clearly $M^{*,Z^1}x = M^*x = M^{Z^1,*}x$, for all individual variables $x \in \mathcal{V}_0$.

2. Let $X^1 \in \mathcal{V}_1$. If $X^1 \notin \{Z_1^1, \dots, Z_m^1\}$, then

$$M^{Z^1,*}X^1 = M^{Z^1}X^1 \cap D^* = MX^1 \cap D^* = M^*X^1 = M^{*,Z^1}X^1 \,.$$

   On the other hand, if $X^1 = Z_j^1$ for some $j \in \{1, \dots, m\}$, we have

$$M^{Z^1,*}Z_j^1 = M^{Z^1}Z_j^1 \cap D^* = U_j^1 \cap D^* = U_j^1 = M^{*,Z^1}Z_j^1 \,.$$

3. Let $X^2 \in \mathcal{V}_2$. Then we have

$$\begin{aligned} M^{*,Z^1}X^2 = M^*X^2 = &((MX^2 \cap \mathrm{pow}(D^*)) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\cup \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F),\ MX^1 \in MX^2\}\,, \end{aligned}$$

$$\begin{aligned} M^{Z^1,*}X^2 &= ((M^{Z^1}X^2 \cap \mathrm{pow}(D^*)) \setminus \{M^{Z^1,*}X^1 : X^1 \in ((\mathcal{V}_1' \cup \mathcal{V}_1^F) \cup \{Z_1^1, \dots, Z_m^1\})\}) \\ &\quad \cup\{M^{Z^1,*}X^1 : X^1 \in ((\mathcal{V}_1' \cup \mathcal{V}_1^F) \cup \{Z_1^1, \dots, Z_m^1\}),\ M^{Z^1}X^1 \in M^{Z^1}X^2\} \\ &= ((MX^2 \cap \mathrm{pow}(D^*)) \setminus (\{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\} \cup \{U_j : j = 1, \dots, m\})) \\ &\quad \cup(\{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F),\ MX^1 \in MX^2\} \\ &\quad \cup(\{U_j : j = 1, \dots, m\} \cap MX^2))\,. \end{aligned}$$

   By putting

$$\begin{aligned} &P_1 = MX^2 \cap \mathrm{pow}(D^*), \\ &P_2 = \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}, \\ &P_3 = \{U_j : j = 1, \dots, m\}, \\ &P_4 = \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F),\ MX^1 \in MX^2\}, \\ &P_5 = \{U_j : j = 1, \dots, m\} \cap MX^2, \end{aligned}$$

   the above relations can be rewritten as

$$\begin{aligned} M^{*,Z^1}X^2 &= (P_1 \setminus P_2) \cup P_4 \\ M^{Z^1,*}X^2 &= (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5\,. \end{aligned}$$

   Moreover, it is easy to verify that the following relations hold:

$$\begin{aligned} &P_2 \cap P_3 = \emptyset \\ &P_5 = P_1 \cap P_3 \\ &P_4 \subseteq P_2\,. \end{aligned}$$

   Therefore we have

$$\begin{aligned} (P_1 \setminus P_2) \cup P_4 &= (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup (P_1 \cap P_3) \\ &= (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5 \end{aligned}$$

   i.e., we have $M^{*,Z^1}X^2 = M^{Z^1,*}X^2$.

4. Let $X^3 \in \mathcal{V}_3$, then $M^{*,Z^1}X^3 = M^*[Z_1^1/U_1^1, \dots, Z_m^1/U_m^1]X^3 = M^*X^3$ and

$$\begin{aligned} M^{Z^1,*}X^3 &= ((M^{Z^1}X^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^{Z^1,*}X^2 : X^2 \in \mathcal{V}_2'\}) \\ &\quad \cup\{M^{Z^1,*}X^2 : X^2 \in \mathcal{V}_2', M^{Z^1}X^2 \in M^{Z^1}X^3\} \\ &= ((MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^*X^2 : X^2 \in \mathcal{V}_2'\}) \\ &\quad \cup\{M^*X^2 : X^2 \in \mathcal{V}_2', MX^2 \in MX^3\} \\ &= M^*X^3\,. \end{aligned}$$

   Since $M^{*,Z^1}X^3 = M^{Z^1,*}X^3$ the thesis follows. ■

A.4 Proof of Lemma 4

Lemma 4. *Let* $Z_1^2, \ldots, Z_p^2 \in \mathcal{V}_2 \setminus \mathcal{V}_2'$ *and* $U_1^2, \ldots, U_p^2 \in \mathrm{pow}(\mathrm{pow}(D^*)) \setminus \{M^*X^2 : X^2 \in \mathcal{V}_2'\}$. *Then the* $4LQS^R$*-interpretations* $\boldsymbol{\mathcal{M}}^{*,Z^2}$ *and* $\boldsymbol{\mathcal{M}}^{Z^2,*}$ *coincide.*

Proof. We show that $\mathcal{M}^{*,Z^2}$ and $\mathcal{M}^{Z^2,*}$ coincide by proving that they agree over variables of all sorts.

1. Plainly $M^{*,Z^2}x = M^*x = M^{Z^2,*}x$, for every $x \in \mathcal{V}_0$.
2. Let $X^1 \in \mathcal{V}_1$, then $M^{*,Z^2}X^1 = M^*X^1 = M^{Z^2,*}X^1$.
3. Let $X^2 \in \mathcal{V}_2$ such that $X^2 \notin \{Z_1^2, \ldots, Z_p^2\}$, then

$$M^{*,Z^2}X^2 = M^*[Z_1^2/U_1^2, \ldots, Z_p^2/U_p^2]X^2 = M^*X^2,$$

and

$$\begin{aligned} M^{Z^2,*}X^2 &= ((M^{Z^2}X^2 \cap \mathrm{pow}(D^*)) \setminus \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\qquad \cup \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F), M^{Z^2}X^1 \in M^{Z^2}X^2\} \\ &= ((MX^2 \cap pow(D^*)) \setminus \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\qquad \cup \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F), MX^1 \in MX^2\} \\ &= M^*X^2 \,. \end{aligned}$$

Since $M^{*,Z^2}X^2 = M^{Z^2,*}X^2$ the thesis follows. On the other hand, if $X^2 \in \{Z_1^2, \ldots, Z_p^2\}$, say $X^2 = Z_j^2$, then $M^{*,Z^2}X^2 = U_j^2$, and

$$\begin{aligned} M^{Z^2,*}X^2 &= ((M^{Z^2}X^2 \cap \mathrm{pow}(D^*)) \setminus \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\qquad \cup \{M^{Z^2,*}X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F), M^{Z^2}X^1 \in M^{Z^2}X^2\} \\ &= (U_j^2 \setminus \{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F)\}) \\ &\qquad \cup (\{M^*X^1 : X^1 \in (\mathcal{V}_1' \cup \mathcal{V}_1^F), MX^1 \in U_j^2\}) \\ &= U_j^2 \,. \end{aligned}$$

Clearly the thesis follows also in this case.

4. Let $X^3 \in \mathcal{V}_3$. Then we have

$$\begin{aligned} M^{*,Z^2}X^3 = M^*X^3 &= ((MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^*X^2 : X^2 \in \mathcal{V}_2'\}) \\ &\qquad \cup \{M^*X^2 : X^2 \in \mathcal{V}_2',\ MX^2 \in MX^3\} \\ M^{Z^2,*}X^3 &= ((M^{Z^2}X^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus \{M^{Z^2,*}X^2 : X^2 \in \mathcal{V}_2' \cup \{Z_1^2, \ldots, Z_p^2\}\}) \\ &\qquad \cup \{M^{Z^2,*}X^2 : X^2 \in \mathcal{V}_2' \cup \{Z_1^2, \ldots, Z_p^2\},\ M^{Z^2}X^2 \in M^{Z^2}X^3\} \\ &= ((MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*))) \setminus (\{M^*X^2 : X^2 \in \mathcal{V}_2'\} \cup \{U_j^2 : j = 1, \ldots, p\})) \\ &\qquad \cup \{M^*X^2 : X^2 \in \mathcal{V}_2',\ MX^2 \in MX^3\} \cup (\{U_j^2 : j = 1, \ldots, p\} \cap MX^3) \,. \end{aligned}$$

By putting

$$\begin{aligned} P_1 &= MX^3 \cap \mathrm{pow}(\mathrm{pow}(D^*)) \\ P_2 &= \{M^*X^2 : X^2 \in \mathcal{V}_2'\} \\ P_3 &= \{U_j^2 : j = 1, \ldots, p\} \\ P_4 &= \{M^*X^2 : X^2 \in \mathcal{V}_2',\ MX^2 \in MX^3\} \\ P_5 &= \{U_j^2 : j = 1, \ldots, p\} \cap MX^3 \end{aligned}$$

then the above relations can be rewritten as

$$
\begin{aligned}
M^{*,Z^2}X^3 &= (P_1 \setminus P_2) \cup P_4 \\
M^{Z^2,*}X^3 &= (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5 \,.
\end{aligned}
$$

Moreover, it is easy to verify that the following relations hold:

$$
\begin{aligned}
&P_2 \cap P_3 = \emptyset \\
&P_5 = P_1 \cap P_3 \\
&P_4 \subseteq P_2 \,.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
(P_1 \setminus P_2) \cup P_4 &= (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup (P_1 \cap P_3) \\
&= (P_1 \setminus (P_2 \cup P_3)) \cup P_4 \cup P_5
\end{aligned}
$$

i.e., we have $M^{*,Z^2}X^3 = M^{Z^2,*}X^3$. ■

A.5 Proof of Lemma 7

Lemma 7. *For every formula φ of the logic* S5*, φ is satisfiable in a model $K = \langle W, R, h\rangle$ iff there is a $4LQS^R$-interpretation satisfying $x \in X_\varphi$.*

Proof. Let $\bar{w}$ be a world in W. We construct a $4LQS^R$-interpretation $\boldsymbol{\mathcal{M}} = (W, M)$ as follows:

- $Mx = \bar{w}$,
- $MX^1_p = h(p)$, where p is a propositional letter and $X^1_p = \tau_{\mathsf{S5}}(p)$,
- $M\tau_{\mathsf{S5}}(\psi) = \textbf{true}$, for every $\psi \in SubF(\varphi)$, where ψ is not a propositional letter.

To prove the lemma, it would be enough to show that $K, \bar{w} \models \varphi$ iff $M \models x \in X^1_\varphi$. However, it is more convenient to prove the following more general property:

Given a $w \in W$, if $y \in \mathcal{V}_0$ is such that $My = w$, then

$$K, w \models \varphi \textit{ iff } M \models y \in X^1_\varphi,$$

which we do by structural induction on φ.

Base case: If φ is a propositional letter, by definition, $K, w \models \varphi$ iff $w \in h(\varphi)$. But this holds iff $My \in MX^1_\varphi$, which is equivalent to $M \models y \in X^1_\varphi$.

Inductive step: We consider only the cases in which $\varphi = \Box\psi$ and $\varphi = \Diamond\psi$, as the other cases can be dealt with similarly.

- If $\varphi = \Box\psi$, assume first that $K, w \models \Box\psi$. Then $K, w \models \psi$ and, by inductive hypothesis, $M \models y \in X^1_\psi$. Since $M \models \tau_{\mathsf{S5}}(\Box\psi)$, it holds that $M \models (\forall z_1)(z_1 \in X^1_\psi) \rightarrow (\forall z_2)(z_2 \in X^1_{\Box\psi})$. Then we have $M[z_1/w, z_2/w] \models (z_1 \in X^1_\psi) \rightarrow (z_2 \in X^1_{\Box\psi})$ and, since $My = w$, we have also that $M \models (y \in X^1_\psi) \rightarrow (y \in X^1_{\Box\psi})$. By the inductive hypothesis and by modus ponens we obtain $M \models y \in X^1_{\Box\psi}$, as required.

On the other hand, if $K, w \not\models \Box\psi$, then $K, w \not\models \psi$ and, by inductive hypothesis, $M \not\models y \in X^1_\psi$. Since $M \models \tau_{\mathsf{S5}}(\Box\psi)$, then $M \models \neg(\forall z_1)(z_1 \in X^1_\psi) \rightarrow (\forall z_2)\neg(z_2 \in X^1_{\Box\psi})$. By the inductive hypothesis and some predicate logic manipulations, we have $M \models \neg(y \in X^1_\psi) \rightarrow \neg(y \in X^1_{\Box\psi})$, and by modus ponens we infer $M \models \neg(y \in X^1_{\Box\psi})$, as we wished to prove.

- Let $\varphi = \Diamond\psi$ and, to begin with, assume that $K, w \models \Diamond\psi$. Then, there is a w' such that $K, w' \models \psi$, and a $y' \in \mathcal{V}_0$ such that $My' = w'$. Thus, by inductive hypothesis, we have $M \models y' \in X^1_\psi$ and, by predicate logic, $M \models \neg(\forall z_1)\neg(z_1 \in X^1_\psi)$. By the very definition of M, $M \models \tau_{\mathsf{S5}}(\Diamond\psi)$ and thus $M \models \neg(\forall z_1)\neg(z_1 \in X^1_\psi) \rightarrow (\forall z_2)(z_2 \in X^1_{\Diamond\psi})$. Then, by modus ponens we obtain $M \models (\forall z_2)(z_2 \in X^1_{\Diamond\psi})$ and finally, by predicate logic, $M \models y \in X^1_{\Diamond\psi}$.

  On the other hand, if $K, w \not\models \Diamond\psi$, then $K, w' \not\models \psi$, for any $w' \in W$ and, since $w' = My'$ for any $y' \in \mathcal{V}_0$, it holds that $M \not\models y' \in X^1_\psi$ and thus, by predicate logic, $M \models (\forall z_1)\neg(z_1 \in X^1_\psi)$.

  Reasoning as above, $M \models (\forall z_1)\neg(z_1 \in X^1_\psi) \rightarrow (\forall z_2)\neg(z_2 \in X^1_{\Diamond\psi})$ and, by modus ponens, $M \models (\forall z_2)\neg(z_2 \in X^1_{\Diamond\psi})$. Finally, by predicate logic, $M \not\models y \in X^1_{\Diamond\psi}$, as required. ∎

A.6 Proof of Lemma 8

Lemma 8. *For every formula φ of the logic τ_{K45}, φ is satisfiable in a model $K = \langle W, R, h\rangle$ iff there is a $4LQS^R$-interpretation satisfying $x \in X_\varphi$.*

Proof. We proceed as in the proof of Lemma 7, by constructing a $4LQS^R$-interpretation $\boldsymbol{\mathcal{M}} = (W, M)$ which has the following property:

Given a $w \in W$ and a $y \in \mathcal{V}_0$ such that $My = w$, it holds that

$$K, w \models \varphi \text{ iff } M \models y \in X^1_\varphi.$$

We proceed by structural induction on φ. As with Lemma 7, we consider only the cases in which $\varphi = \Box\psi$ and $\varphi = \Diamond\psi$.

- Let $\varphi = \Box\psi$ and assume that $K, w \models \Box\psi$. Let v be a world of W such that there is a $u \in W$ with $\langle u, v\rangle \in R^3$, and let $x_1, x_2 \in \mathcal{V}_0$ be such that $v = Mx_1$ and $u = Mx_2$. We have that $K, v \models \psi$ and, by inductive hypothesis, $M \models x_1 \in X^1_\psi$. Since $M \models \tau_{\mathsf{K45}}(\Box\psi)$, then $M \models (\forall z_1)((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \rightarrow z_1 \in X^1_\psi) \rightarrow (\forall z)(z \in X^1_{\Box\psi})$. Hence $M[z_1/v, z_2/u, z/w] \models (\langle z_2, z_1\rangle \in R^3 \rightarrow z_1 \in X^1_\psi) \rightarrow z \in X^1_{\Box\psi}$ and thus $M \models (\langle x_2, x_1\rangle \in R^3 \rightarrow x_1 \in X^1_\psi) \rightarrow y \in X^1_{\Box\psi}$. Since $M \models \langle x_2, x_1\rangle \in R^3 \rightarrow x_1 \in X^1_\psi$, by modus ponens we have the thesis. The thesis follows also in the case in which there is no u such that $\langle u, v\rangle \in R^3$. In fact, in that case $M \models \langle x_2, x_1\rangle \in R^3 \rightarrow x_1 \in X^1_\psi$ holds for any $x_2 \in \mathcal{V}_0$.

Consider next the case in which $K, w \not\models \Box\psi$. Then, there must be a $v \in W$ such that there is a u with $\langle u, v\rangle \in R^3$ and $K, v \not\models \psi$. Let $x_1, x_2 \in \mathcal{V}_0$ be such that $Mx_1 = v$ and $Mx_2 = u$. Then, by inductive hypothesis, $M \not\models x_1 \in X^1_\psi$.
By definition of M, we have $M \models \neg(\forall z_1)\neg((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \wedge \neg(z_1 \in X^1_\psi)) \to (\forall z)\neg(z \in X^1_{\Box\psi})$. By the above instantiations and by the hypotheses, we have that $M \models ((\langle x_2, x_1\rangle \in R^3) \wedge \neg(x_1 \in X^1_\psi)) \to \neg(y \in X^1_{\Box\psi})$ and $M \models (\langle x_2, x_1\rangle \in R^3) \wedge \neg(x_1 \in X^1_\psi)$. Thus, by modus ponens, we obtain the thesis.

- Let $\varphi = \Diamond\psi$ and assume that $K, w \models \Diamond\psi$. Then there are $u, v \in W$ such that $\langle u, v\rangle \in R$ and $K, v \models \psi$. Let $x_1, x_2 \in \mathcal{V}_0$ be such that $Mx_1 = v$ and $Mx_2 = u$. Then, by inductive hypothesis, $M \models x_1 \in X^1_\psi$. Since $M \models \tau_{\mathsf{K45}}(\Diamond\psi)$, it follows that $M \models \neg(\forall z_1)\neg((\neg(\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \wedge z_1 \in X^1_\psi) \to (\forall z)(z \in X^1_{\Diamond\psi})$. By the hypotheses and the variable instantiations above it follows that $M \models ((\langle x_2, x_1\rangle \in R^3) \wedge x_1 \in X^1_\psi) \to y \in X^1_{\Diamond\psi}$ and $M \models (\langle x_2, x_1\rangle \in R^3) \wedge x_1 \in X^1_\psi$. Finally, by an application of modus ponens the thesis follows.
  On the other hand, if $K, w \not\models \Diamond\psi$, then for every $v \in W$, either there is no $u \in W$ such that $\langle u, v\rangle \in R$, or $K, v \not\models \psi$. Let $x_1, x_2 \in \mathcal{V}_0$ be such that $Mx_1 = v$ and $Mx_2 = u$. If $K, v \not\models \psi$, by inductive hypothesis, we have that $M \not\models y \in X^1_\psi$.
  Since $M \models (\forall z_1)(((\forall z_2)\neg(\langle z_2, z_1\rangle \in R^3)) \vee \neg(z_1 \in X^1_\psi)) \to (\forall z)\neg(z \in X^1_{\Diamond\psi})$, by the hypotheses and by the variable instantiations above we get $M \models (\neg(\langle x_2, x_1\rangle \in R^3) \vee \neg(x_1 \in X^1_\psi)) \to \neg(y \in X^1_{\Diamond\psi})$ and $M \models (\neg(\langle x_2, x_1\rangle \in R^3) \vee \neg(x_1 \in X^1_\psi))$. Finally, by modus ponens we infer the thesis. ■