On the Approximation of Mean-Payoff Games

Raffaella Gentilini

University of Perugia

Convegno Italiano Logica Computazionale (CILC2011)
Contents

1. Mean-Payoff Games (MPG) Problems
2. Exact Solutions for MPG
3. Approximate Solutions for MPG: The Additive Setting
4. Approximate Solutions for MPG: The Multiplicative Setting
Mean-Payoff Games MPG

- 2 players:
  Maximazer Bob vs Minimazer Alice

- Played on a finite graph (arena)
- Turn based
- Infinite number of turns
- Goal (for Bob): maximizing the long-run average weight
Mean-Payoff Games MPG

- 2 players: 
  Maximazer Bob vs Minimazer Alice

- played on a finite graph (arena)
Mean-Payoff Games MPG

- 2 players:
  Maximazer $\Box$ Bob vs Minimazer $\triangle$ Alice

- played on a finite graph (arena)
- turn based
Mean-Payoff Games MPG

- 2 players: Maximazer Bob vs Minimazer Alice
- played on a finite graph (arena)
- turn based
- infinite number of turns
Mean-Payoff Games MPG

- 2 players: Maximazer Bob vs Minimazer Alice

- played on a finite graph (arena)
- turn based
- infinite number of turns
- goal (for Bob): maximizing the long-run average weight
Mean-Payoff Games

Exact Solutions

Approximation

MPG in Formal Term

In a MPG $\Gamma = (V, E, w : V \rightarrow \mathbb{Z}, \langle V_{\Box}, V_{\triangle} \rangle)$:

Bob wants to maximize his payoff, i.e. the long-run average weight in a play.

Given a play $p = \{v_i\}_{i \in \mathbb{N}}$ in $\Gamma$, the payoff of Bob on $p$ is:

$$\text{MP}(v_0 v_1 \ldots v_n \ldots) = \lim_{n \to \infty} \inf \frac{1}{n} \cdot \sum_{i=0}^{n-1} w(v_i, v_{i+1})$$
The value secured by a strategy $\sigma_{\Box}$: $V^* \cdot V_{\Box} \rightarrow V$ in vertex $v$ is:

$$\text{val}^{\sigma_{\Box}}(v) = \inf_{\sigma_{\triangle} \in \Sigma_{\triangle}} \text{MP(outcome}^\Gamma(v, \sigma_{\Box}, \sigma_{\triangle}))$$

$$\sup_{\sigma_{\Box} \in \Sigma_{\Box}}(\text{val}^{\sigma_{\Box}}(v))$$ is the optimal value that B\text{\textcopyright}b can secure in $v$.
The value secured by a strategy $\sigma_{\Box}$: $V^* \cdot V_{\Box} \rightarrow V$ in vertex $v$ is:

$$\text{val}^{\sigma_{\Box}}(v) = \inf_{\sigma_{\triangle} \in \Sigma_{\triangle}} \text{MP}(\text{outcome}^\Gamma(v, \sigma_{\Box}, \sigma_{\triangle}))$$

$$\sup_{\sigma_{\Box} \in \Sigma_{\Box}} (\text{val}^{\sigma_{\Box}}(v))$$ is the optimal value that $B_{\Box}b$ can secure in $v$.
Mean-Payoff Games

Exact Solutions

Approximation

MPG in Formal Term

Example

\[ \text{val}^{\sigma \Box}(v) = \frac{-4}{2} \]

The value secured by a strategy \( \sigma \Box \): \( V^* \cdot V_{\Box} \rightarrow V \) in vertex \( v \) is:

\[ \text{val}^{\sigma \Box}(v) = \inf_{\sigma \Delta \in \Sigma_{\Delta}} \text{MP(outcome}_{\Gamma}(v, \sigma \Box, \sigma \Delta)) \]

\[ \sup_{\sigma \Box \in \Sigma_{\Box}} (\text{val}^{\sigma \Box}(v)) \] is the optimal value that \( B_{\Box} b \) can secure in \( v \)
**MPG in Formal Term**

**Example**

The value secured by a strategy \( \sigma \): \( V^* \cdot V \rightarrow V \) in vertex \( v \) is:

\[
\text{val}^{\sigma}(v) = \inf_{\sigma_\Delta \in \Sigma_\Delta} \text{MP(outcome}^\Gamma(v, \sigma, \sigma_\Delta))
\]

\[
\sup_{\sigma \in \Sigma} (\text{val}^{\sigma}(v)) \text{ is the optimal value that B can secure in } v
\]
Mean-Payoff Games

MPG in Formal Term

Example

\[ \sup_{\sigma_\Box \in \Sigma_\Box} (\text{val}^{\sigma_\Box}(v)) = \frac{2}{2} \]

The value secured by a strategy \( \sigma_\Box \): \( V^* \cdot V_\Box \rightarrow V \) in vertex \( v \) is:

\[ \text{val}^{\sigma_\Box}(v) = \inf_{\sigma_\triangle \in \Sigma_\triangle} \text{MP(outcome}_\Gamma(v, \sigma_\Box, \sigma_\triangle)) \]

\[ \sup_{\sigma_\Box \in \Sigma_\Box} (\text{val}^{\sigma_\Box}(v)) \] is the optimal value that \( B_\Box b \) can secure in \( v \)
MPG are Memoryless Determined

**Theorem [Ehrenfeucht&Mycielsky’79]**

\[ \text{val}^\Gamma(v) = \sup_{\sigma\Box \in \Sigma\Box} \inf_{\sigma\Delta \in \Sigma\Delta} \text{MP}(\text{outcome}^\Gamma(v, \sigma\Box, \sigma\Delta)) = \]
\[ = \inf_{\sigma\Delta \in \Sigma\Delta} \sup_{\sigma\Box \in \Sigma\Box} \text{MP}(\text{outcome}^\Gamma(v, \sigma\Box, \sigma\Box)). \]

There exist uniform memoryless strategies, \( \pi\Box : V\Box \to V \) for B\Box b, \( \pi\Delta : V\Delta \to V \) for \( \Delta \)lice such that:

\[ \text{val}^\Gamma(v) = \text{val}^{\pi\Box}(v) = \text{val}^{\pi\Delta}(v). \]
MPG are Memoryless Determined

Example

\[ \text{Theorem [Ehrenfeucht & Mycielsky’79]} \]

\[ \text{val}^\Gamma (v) = \sup_{\sigma_\square \in \Sigma_\square} \text{inf}_{\sigma_\triangle \in \Sigma_\triangle} \text{MP(outcome}^\Gamma (v, \sigma_\square, \sigma_\triangle)) = \]

\[ = \inf_{\sigma_\triangle \in \Sigma_\triangle} \sup_{\sigma_\square \in \Sigma_\square} \text{MP(outcome}^\Gamma (v, \sigma_\square, \sigma_\square)). \]

There exist uniform memoryless strategies, \( \pi_\square : V_\square \to V \) for B\( \square \)b, \( \pi_\triangle : V_\triangle \to V \) for \( \triangle \)lice such that:

\[ \text{val}^\Gamma (v) = \text{val}^{\pi_\square} (v) = \text{val}^{\pi_\triangle} (v). \]
MPG are Memoryless Determined

Example

\[
\begin{align*}
\text{val} \Gamma(v) &= \sup_{\sigma \square \in \Sigma \square} \inf_{\sigma \triangle \in \Sigma \triangle} \text{MP}(\text{outcome} \Gamma(v, \sigma \square, \sigma \triangle)) = \\
&= \inf_{\sigma \triangle \in \Sigma \triangle} \sup_{\sigma \square \in \Sigma \square} \text{MP}(\text{outcome} \Gamma(v, \sigma \square, \sigma \square)).
\end{align*}
\]

There exist uniform memoryless strategies, \( \pi \square : V \square \to V \) for \( B \square b \), \( \pi \triangle : V \triangle \to V \) for \( \triangle \)lic\( e \) such that:

\[
\text{val} \Gamma(v) = \text{val}^{\pi \square}(v) = \text{val}^{\pi \triangle}(v).
\]
Mean-Payoff Games

MPG are Memoryless Determined

Example

\[
\Gamma(v) = \frac{\alpha}{\delta} \in \mathbb{Q} : 0 \leq \delta \leq |V| \text{ and } \frac{|\alpha|}{\delta} \leq M = \max_{e \in E} \{|w(e)|\}.
\]

**Theorem [Ehrenfeucht & Mycielsky’79]**

\[
\Gamma(v) = \sup_{\sigma_\Box \in \Sigma_\Box} \inf_{\sigma_\triangle \in \Sigma_\triangle} \text{MP}(\text{outcome}_{\Gamma}(v, \sigma_\Box, \sigma_\triangle)) = \inf_{\sigma_\triangle \in \Sigma_\triangle} \sup_{\sigma_\Box \in \Sigma_\Box} \text{MP}(\text{outcome}_{\Gamma}(v, \sigma_\Box, \sigma_\Box)).
\]

There exist uniform memoryless strategies, \( \pi_\Box : V_\Box \rightarrow V \) for B\( \Box \)b , \( \pi_\triangle : V_\triangle \rightarrow V \) for \( \triangle \)lice such that:

\[
\Gamma(v) = \frac{\alpha}{\delta} \in \mathbb{Q} : 0 \leq \delta \leq |V| \text{ and } \frac{|\alpha|}{\delta} \leq M = \max_{e \in E} \{|w(e)|\}.
\]
Mean-Payoff Games

Exact Solutions

Approximation

MPG Problems

1. Decision Problem Given \( v \in V, \mu \in \mathbb{Z} \), decide if \( B \square b \) has a strategy \( \pi \square \) to secure \( \text{val} \pi \square (v) \geq \mu \).

2. Value Problem: Compute the set of (rational) values:
\[ \left\{ \text{val} \Gamma (v) \mid v \in V \right\} \]

3. (Optimal) Strategy Synthesis Construct an (optimal) strategy for \( B \square b \).
MPG Problems

1. Decision Problem: Given $v \in V$, $\mu \in \mathbb{Z}$, decide if Bob has a strategy $\pi$ to secure $\text{val}^{\pi}(v) \geq \mu$. 
MPG Problems

1. **Decision Problem**
   Given \( v \in V, \mu \in \mathbb{Z} \), decide if \( B \Box b \) has a strategy \( \pi^\Box \) to secure \( \text{val}^{\pi^\Box}(v) \geq \mu \).

2. **Value Problem**: Compute the set of (rational) values:

\[
\{ \text{val}^\Gamma(v) \mid v \in V \}
\]
MPG Problems

1. **Decision Problem** Given $v \in V$, $\mu \in \mathbb{Z}$, decide if $B \Box b$ has a strategy $\pi_B$ to secure $\text{val}^{\pi_B}(v) \geq \mu$.

2. **Value Problem**: Compute the set of (rational) values:

   \[ \{ \text{val}^\Gamma(v) \mid v \in V \} \]

3. **(Optimal) Strategy Synthesis** Construct an (optimal) strategy for $B \Box b$. 

MPG Problems: Why They Matter?

Correctness Relation

Quantitative Requirements:

- limited resources
- average performance ...
Mean-Payoff Games

MPG Problems: Why They Matter?

System Model?

Correctness Relation

Quantitative Requirements:
- limited resources
- average performance . . .
MPG Problems: Why They Matter?

Solved as a game: System vs Environment
Solution = Winning Strategy

System Model?

Correctness Relation

Quantitative Requirements:
- limited resources
- average performance ...
MPG Problems: Why They Matter?

- MPG significative for theoretical and applicative aspects
  - $\mu$-calculus model checking $\Leftrightarrow$ parity games $\Rightarrow$ MPG
  - MPG $\Rightarrow$ simple stochastic games
  - MPG $\Rightarrow$ discounted payoff games

No polynomial algorithm known so far.
MPG Problems: Why They Matter?

- MPG significative for theoretical and applicative aspects
  - $\mu$-calculus model checking $\Leftrightarrow$ parity games $\Rightarrow$ MPG
  - MPG $\Rightarrow$ simple stochastic games
  - MPG $\Rightarrow$ discounted payoff games
- MPG problems have an interesting complexity status
  - MPG decision problem belongs to $\text{NP} \cap \text{coNP}$ (and even to $\text{UP} \cap \text{coUP}$)
  - No polynomial algorithm known so far
Solving MPG Problems

Consider $\Gamma = (V, E, w, \langle V_\square, V_\triangle \rangle)$, where $w : V \rightarrow [-M \cdots + M]$: 

$\Theta(\sqrt{EV^2M})$ algorithm for the decision problem

$\Theta(\sqrt{EV^3M})$ algorithm for the value problem

$\Theta(\sqrt{EV^4M \log(VE^2)})$ algorithm for optimal strategy synthesis

$O(\sqrt{EV^2V \log(Z)})$ algorithm for the decision/value problem

U. Zwick and M. Paterson, 1996

Y. Lifshits and D. Pavlov, 2006
Solving MPG Problems

Consider $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$, where $w : V \rightarrow [-M \cdot \cdot + M]$:

U. Zwick and M. Paterson, 1996

- $\Theta(EV^2M)$ algorithm for the decision problem
- $\Theta(EV^3M)$ algorithm for the value problem
- $\Theta(EV^4M \log(E/V))$ algorithm for optimal strategy synthesis

Y. Lifshits and D. Pavlov, 2006

- $O(EV^2V \log(Z))$ algorithm for the decision/value problem
Solving MPG Problems

Consider $\Gamma = (V, E, w, \langle V_\square, V_\triangle \rangle)$, where $w : V \rightarrow [-M \cdots + M]$:

- $\Theta(EV^2 M)$ algorithm for the decision problem

H. Bjorklund and S. Vorobyov, 2004: Use a randomized framework

- $O(\min(EV^2 M, 2^{O(\sqrt{V \log V})}))$ for the decision prob.
- $O(\min(EV^3 M(\log V + \log M), 2^{O(\sqrt{V \log V})}))$ for the value prob.
- $\Theta(EV^4 M \log(\frac{E}{V}))$ algorithm for optimal strategy synthesis

U. Zwick and M. Paterson, 1996

- $\Theta(EV^2 M)$ algorithm for the decision problem
Solving MPG Problems

Consider $\Gamma = (V, E, w, \langle V_{\square}, V_{\triangle} \rangle)$, where $w : V \rightarrow [-M \cdots + M]$:

U. Zwick and M. Paterson, 1996

- $\Theta(EV^2M)$ algorithm for the decision problem

H. Bjorklund and S. Vorobyov, 2004: Use a randomized framework

- $O(\min(EV^2M, 2^{O(\sqrt{V \log V})}))$ for the decision prob.
- $O(\min(EV^3M(\log V + \log M), 2^{O(\sqrt{V \log V})}))$ for the value prob.
- $\Theta(EV^4M \log(\frac{E}{V}))$ algorithm for optimal strategy synthesis

Y. Lifshits and D. Pavlov, 2006

- $O(EV^2V \log(Z))$ algorithm for the decision/value problem
Solving MPG Problems

Consider $\Gamma = (V, E, w, \langle V_{\square}, V_{\triangle} \rangle)$, where $w : V \rightarrow [-M \cdots + M]$

U. Zwick and M. Paterson, 1996

- $\Theta(EV^2M)$ algorithm for the decision problem
- $\Theta(EV^3M)$ algorithm for the value problem
- $\Theta(EV^4M \log(\frac{E}{V}))$ algorithm for optimal strategy synthesis

Pseudopolynomial Algorithms


- $O(E \cdot V \cdot M)$ for the decision problem & strategy synthesis
- $O(E \cdot V^2 \cdot M \cdot (\log(V) + \log(M)))$ for the value problem
- $O(E \cdot V^2 \cdot M \cdot (\log(V) + \log(M)))$ algorithm for optimal strategy synthesis
Value Approximation: Basics (I)

Let $\Gamma = (V, E, w, \langle V_0, V_1 \rangle)$ be a MPG, let $v \in V$ and consider $\varepsilon \geq 0$.

**Definition (MPG additive $\varepsilon$-value)**

The value $\tilde{val} \in \mathbb{Q}$ is said an *additive $\varepsilon$-value* on $v$ if and only if:

$$|\tilde{val} - val_{\Gamma}(v)| \leq \varepsilon$$

**Definition (MPG relative $\varepsilon$-value)**

The value $\tilde{val} \in \mathbb{Q}$ is said an *relative $\varepsilon$-value* on $v$ if and only if:

$$\frac{|\tilde{val} - val_{\Gamma}(v)|}{|val_{\Gamma}(v)|} \leq \varepsilon$$
Value Approximation: Basics (II)

Let $\Gamma = (V, E, w, \langle V_0, V_1 \rangle)$ be a MPG:

**MPG Polynomial Time Approximation Scheme (PTAS)**

An additive/relative polynomial approximation scheme for $\Gamma$ is an algorithm $A$ such that for all $\varepsilon > 0$, $A$ computes an additive/relative $\varepsilon$-value in time polynomial w.r.t. the size of $\Gamma$. 
Value Approximation: Basics (II)

Let $\Gamma = (V, E, w, \langle V_0, V_1 \rangle)$ be a MPG:

**MPG Polynomial Time Approximation Scheme (PTAS)**

An additive/relative polynomial approximation scheme for $\Gamma$ is an algorithm $A$ such that for all $\varepsilon > 0$, $A$ computes an additive/relative $\varepsilon$-value in time polynomial w.r.t. the size of $\Gamma$.

**MPG Fully Polynomial Time Approximation Scheme (FPTAS)**

An additive/relative fully polynomial approximation scheme for $\Gamma$ is an algorithm $A$ such that for all $\varepsilon > 0$, $A$ computes an additive/relative $\varepsilon$-value in time polynomial w.r.t. $\Gamma$ and $\frac{1}{\varepsilon}$.
Additive Approximations – FPTAS (I)

A. Roth, M. Balcan, A. Kalai & Y. Mansour – 2010

The MPGvalue problem on graphs with rational weights in \([-1,+1]\) admits an additive FPTAS.
Additive Approximations – FPTAS (I)

The MPGvalue problem on graphs with rational weights in $[-1,+1]$ admits an additive FPTAS.

Easy approximation algorithm based on:
existing pseudopolynomial procedures + graph reweighting
Additive Approximations – FPTAS (II)

Can we efficiently approximate the value in MPG with no restriction on the weights?

**Theorem**

The MPG value problem does not admit an additive FPTAS, unless it is in P.

**Proof (Sketch):**

• By contradiction. Assume an additive FPTAS exists.
• Choose $\varepsilon = \frac{1}{2^n(n-1)}$ and compute the additive $\varepsilon$-value $v_{\varepsilon}$.
• The MPG value $v$ is the unique rational with denominator $1 \leq d \leq n$ in the interval $[v_{\varepsilon} - \varepsilon, v_{\varepsilon} + \varepsilon]$. 

Additive Approximations – FPTAS (II)

Can we efficiently approximate the value in MPG with no restriction on the weights?

Theorem

The MPG value problem does not admit an additive FPTAS, unless it is in P.
Can we efficiently approximate the value in MPG with no restriction on the weights?

**Theorem**

The MPG value problem does not admit an additive FPTAS, unless it is in P.

**Proof (Sketch):**

• By contradiction. Assume an additive FPTAS exists.

  • Choose $\varepsilon = \frac{1}{2}^n(n-1)$ and compute the additive $\varepsilon$-value $v_\varepsilon$.

  • The MPG value $v$ is the unique rational with denominator $1 \leq d \leq n$ in the interval $[v_\varepsilon - \varepsilon, v_\varepsilon + \varepsilon]$. 
Can we efficiently approximate the value in MPG with no restriction on the weights?

Theorem

The MPG value problem does not admit an additive FPTAS, unless it is in P.

Proof (Sketch):

- By contradiction. Assume an additive FPTAS exists.
Additive Approximations – FPTAS (II)

Can we efficiently approximate the value in MPG with no restriction on the weights?

Theorem

The MPG value problem does not admit an additive FPTAS, unless it is in P.

Proof (Sketch):

- By contradiction. Assume an additive FPTAS exists.
- Choose \( \varepsilon = \frac{1}{2n(n-1)} \) and compute the additive \( \varepsilon \)-value \( v^\varepsilon \).
Additive Approximations – FPTAS (II)

Can we efficiently approximate the value in MPG with no restriction on the weights?

Theorem

The MPG value problem does not admit an additive FPTAS, unless it is in P.

Proof (Sketch):

- By contradiction. Assume an additive FPTAS exists.
- Choose $\varepsilon = \frac{1}{2n(n-1)}$ and compute the additive $\varepsilon$-value $v^\varepsilon$.
- The MPG value $v$ is the unique rational with denominator $1 \leq d \leq n$ in the interval $[v^\varepsilon - \varepsilon, v^\varepsilon + \varepsilon]$. 
Are weaker notions of approximation useful to obtain some positive result w.r.t. the MPG value approximation problem?
Additive Approximations – PTAS

Are weaker notions of approximation useful to obtain some positive result w.r.t. the MPG value approximation problem?

Theorem

For any constant $k$: If the problem of computing an additive $k$-approximate MPG value can be solved in polynomial time (w.r.t. the size of the MPG), then the MPG value problem belongs to $P$. 

Additive Approximations
Corollary

The following problems are \( P \)-time equivalent:
Additive Approximations

Corollary

The following problems are \( \mathcal{P} \)-time equivalent:

1. Solving the MPG value problem.
Additive Approximations

Corollary

The following problems are P-time equivalent:

1. Solving the MPG value problem.
2. Determining an additive FPTAS for the MPG value problem.
Corollary

The following problems are $P$-time equivalent:

1. Solving the MPG value problem.
2. Determining an additive FPTAS for the MPG value problem.
3. Determining an additive PTAS for the MPG value problem.
Additive Approximations

Corollary

The following problems are \( \mathcal{P} \)-time equivalent:

1. Solving the MPG value problem.
2. Determining an additive FPTAS for the MPG value problem.
3. Determining an additive PTAS for the MPG value problem.
4. Computing an additive \( k \)-approximate MPG value in polynomial time, for any constant \( k \).
Relative Approximations (I)


The MPG value problem on graphs with nonnegative weights admits a relative FPTAS.
Relative Approximations (II)

Can we design efficient relative approximations for the MPG value problem on graphs with no restriction on the weights?
Can we design efficient relative approximations for the MPG value problem on graphs with no restriction on the weights?

**Theorem**

The MPG value problem does not admit a relative PTAS, unless it is in \( P \).
Relative Approximations (III)

Corollary

The following problems are $P$-time equivalent:

1. Solving the MPG value problem.
2. Determining a relative FPTAS for the MPG value problem.
3. Determining a relative PTAS for the MPG value problem.
Relative Approximations (III)

Corollary

The following problems are P-time equivalent:

1. Solving the MPG value problem.
2. Determining a relative FPTAS for the MPG value problem.
3. Determining a relative PTAS for the MPG value problem.
Relative Approximations (III)

Corollary

The following problems are $P$-time equivalent:

1. Solving the MPG value problem.
Relative Approximations (III)

Corollary

The following problems are \( P \)-time equivalent:

1. Solving the MPG value problem.
2. Determining a relative FPTAS for the MPG value problem.
Relative Approximations (III)

Corollary

The following problems are P-time equivalent:

1. Solving the MPG value problem.
2. Determining a relative FPTAS for the MPG value problem.
3. Determining a relative PTAS for the MPG value problem.
The End

Thank you!